Generalizing Monotonicity Inferences to Opposition Inferences

Ka-fat Chow

The Hong Kong Polytechnic University
kfzhouy@yahoo.com

Abstract. This paper generalizes the notion of monotonicities to opposition properties (OPs). Some propositions regarding the OPs of determiners will be proposed and proved. We will also define the notion of OP-chain and deduce a condition that enables us to determine the OPs of an iterated quantifier in its predicates based on the OPs of its constituent determiners.

Keywords: monotonicity inferences, opposition inferences, opposition properties, OP-chain, Generalized Quantifier Theory, Natural Logic

1 Introduction

Van Benthem (2008) characterized monotonicity inferences, an important type of Natural Logic inferences, as "inferences with inclusion premises" of the form "$P \le Q \Rightarrow \varphi(P) \le \varphi(Q)$". He also proposed the study on "inferences with exclusion premises" of the form "$P \le \neg Q \Rightarrow \varphi(P) \le \neg\varphi(Q)$", which we call "opposition inferences". In recent years, some scholars, e.g. MacCartney and Manning (2009), MacCartney (2009), Icard (2011), have started to study opposition inferences as a new type of Natural Logic inferences from different perspectives based on different frameworks. This paper is a study of opposition inferences as a generalization of monotonicity inferences based on the Generalized Quantifier Theory (GQT) framework.

2 Basic Definitions

First of all, let us review the definition of increasing monotonicity in the right argument for type $\langle 1, 1\rangle$ generalized quantifiers (i.e. determiners)[1].

[1] In what follows, we use A and B to denote the left and right argument of a determiner, respectively. We also use the symbol "$\le$" to denote the subset relation between sets as well as the entailment relation between propositions. Note that increasing monotonicity in the left argument can be defined analogously.

Definition 1: Let Q be a determiner, then Q is increasing in the right argument iff for all A, B, B', $B \leq B' \Rightarrow Q(A)(B) \leq Q(A)(B')$.

Note that an equivalent definition can be obtained by replacing both occurrences of "$\leq$" by "$\geq$" in the above definition. Based on Definition 1, we may denote increasing monotonicity figuratively by "$\leq\!\rightarrow\!\leq$" (or equivalently "$\geq\!\rightarrow\!\geq$"). Analogously, decreasing monotonicity may be denoted figuratively by "$\leq\!\rightarrow\!\geq$" (or equivalently "$\geq\!\rightarrow\!\leq$"). Now "$\leq$" and "$\geq$" are just two possible relations between sets / propositions. If we replace them by more general binary relations R_1, R_2 (written in prefix notation), we will obtain a more general definition[2].

Definition 2: Let Q be a determiner, then Q is $R_1 \rightarrow R_2$ in the right argument iff for all A, B, B', $R_1[B, B'] \Rightarrow R_2[Q(A)(B), Q(A)(B')]$.

Hence, increasing and decreasing monotonicites are just special cases of Definition 2 with R_1 and R_2 instantiated as the "inclusion relations", i.e. $\leq$ and $\geq$.

Apart from "inclusion relations", we may also consider "exclusion relations". In this paper, we will consider two "exclusion relations" that are disjunctions of relations in the classical square of opposition: CC (standing for "contrary or contradictory") and SC (standing for "subcontrary or contradictory"). In classical logic, two propositions p and q satisfy the CC relation (denoted CC[p, q] iff they cannot be both true, and they satisfy the SC relation (denoted SC[p, q]) iff they cannot be both false. More generally, we may adopt the following definitions so that these two relations are also applicable to sets: let X and X' be sets or propositions,

$$\mathrm{CC}[X, X'] \Leftrightarrow X \leq \neg X'; \mathrm{SC}[X, X'] \Leftrightarrow \neg X \leq X' \tag{1}$$

For example, we have CC[TEENAGER, ELDERLY] and SC[AGED-OVER-50, AGED-BELOW-51] because an individual cannot be a teenager and an elderly simultaneously, whereas an individual must be either aged over 50 or aged below 51. According to (1), we also have

$$\mathrm{CC}[X, X'] \Leftrightarrow \mathrm{SC}[\neg X, \neg X'] \tag{2}$$

By instantiating R_1 and R_2 in Definition 2 as CC and SC, we then have 4 possible properties of determiners: CC→CC, CC→SC, SC→CC and SC→SC. These 4 properties will henceforth be called "opposition properties" (OPs).

3 OPs of Determiners

Our next task is to classify some commonly used determiners according to their OPs in the two arguments. For convenience, I will denote the sets of determiners possessing or not possessing a certain OP in a certain argument by placing a "+" or "−" sign on the left and right-hand sides of the name of the OP. For

[2] Similar property in the left argument can be defined analogously.

instance, −CC→CC+ denotes the set of those determiners that are CC→CC in the right but not left argument.

In what follows I first state and prove four propositions:

Proposition 1: A determiner Q possesses a certain OP in its right argument iff each of its outer negation (denoted $\neg Q$), (right) inner negation (denoted $Q\neg$) and (right) dual (denoted Q^d)[3] possesses a different OP in its right argument according to the following table:

Q	**¬Q**	**Q¬**	$\mathbf{Q^d}$
CC→CC	CC→SC	SC→CC	SC→SC
CC→SC	CC→CC	SC→SC	SC→CC
SC→CC	SC→SC	CC→CC	CC→SC
SC→SC	SC→CC	CC→SC	CC→CC

Proof: Here we only prove the first row of the table. The remaining rows can be proved similarly. By Definition 2 and (1), $Q \in$ CC→CC+ iff

$$\mathrm{CC}[B, B'] \Rightarrow Q(A)(B) \leq \neg(Q(A)(B')) \tag{3}$$

Now (3) is equivalent to

$$\mathrm{CC}[B, B'] \Rightarrow \neg(\neg Q(A)(B)) \leq \neg Q(A)(B') \tag{4}$$

Substituting the arbitrary sets B and B' by their negations and using (2) and the definitions of (right) inner negation and (right) dual, (3) and (4) can be rewritten as

$$\mathrm{SC}[B, B'] \Rightarrow Q\neg(A)(B) \leq \neg(Q\neg(A)(B')) \tag{5}$$

$$\mathrm{SC}[B, B'] \Rightarrow \neg(Q^d(A)(B)) \leq Q^d(A)(B') \tag{6}$$

From (4)-(6), we have $\neg Q \in$ CC→SC+, $Q\neg \in$ SC→CC+ and $Q^d \in$ SC→SC+. □

Proposition 2: Let Q_1 and Q_2 be determiners such that $Q_1 \leq Q_2$[4].
(a) If Q_2 is CC→CC (SC→CC) in an argument, then so is Q_1 in the same argument.
(b) If Q_1 is CC→SC (SC→SC) in an argument, then so is Q_2 in the same argument.

[3] Outer negation, (right) inner negation and (right) dual are as defined in Peters and Westerståhl (2006).

[4] $Q_1 \leq Q_2$ iff for all A, B, $Q_1(A)(B) \leq Q_2(A)(B)$. From this we can define the following: $Q_1 = Q_2$ iff $Q_1 \leq Q_2$ and $Q_2 \leq Q_1$.

Proof: Here we only prove part (a). Part (b) can be proved similarly. Suppose $Q_2 \in$ CC→CC+ and CC$[B, B']$. Let $\|Q_1(A)(B)\| = 1$. Then since $Q_1 \leq Q_2$, we have $\|Q_2(A)(B)\| = 1$. But then we must have $\|Q_2(A)(B')\| = 0$. By $Q_1 \leq Q_2$ again, we have $\|Q_1(A)(B')\| = 0$. We have thus proved that CC$[B, B'] \Rightarrow$ CC$[Q_1(A)(B), Q_1(A)(B')]$, i.e. $Q_1 \in$ CC→CC+. The proofs for the cases $Q_2 \in$ +CC→CC, SC→CC+ and +SC→CC are exactly the same. □

Proposition 3: Let Q_1 be a symmetric determiner and Q_2 be a contrapositive determiner[5].
(a) Q_1 possesses a certain OP in an argument iff Q_1 possesses the same OP in the other argument.
(b) Q_2 is CC→CC in an argument iff Q_2 is SC→CC in the other argument. Q_2 is CC→SC in an argument iff Q_2 is SC→SC in the other argument.

Proof: Here we only prove part (b). Part (a) can be proved similarly. Suppose $Q_2 \in$ CC→CC+ and SC$[A, A']$, which by (2) is equivalent to CC$[\neg A, \neg A']$. Let $\|Q_2(A)(B)\| = 1$. By the contrapositivity of Q_2, this is equivalent to $\|Q_2(\neg B)(\neg A)\| = 1$. But then we must have $\|Q_2(\neg B)(\neg A')\| = 0$. By the contrapositivity of Q_2 again, this is in turn equivalent to $\|Q_2(A')(B)\| = 0$. We have thus proved that SC$[A, A'] \Rightarrow$ CC$[Q_2(A)(B), Q_2(A')(B)]$, i.e. $Q_2 \in$ +SC→CC. Similarly, we can prove that if $Q_2 \in$ +SC→CC, then $Q_2 \in$ CC→CC+. The proofs for the cases $Q_2 \in$ +CC→CC, CC→SC+ and +CC→SC are exactly the same. □

Proposition 4: On condition that $A \neq \emptyset$[6], (*at least r of*) $\in$ CC→CC+ for $1/2 < r < 1$; (*more than r of*) $\in$ CC→CC+ for $1/2 \leq r < 1$; (*exactly r of*) $\in$ −CC→CC for $0 \leq r \leq 1$.

Proof: Let $\|(\textit{at least } r \textit{ of})(A)(B)\| = 1$ for $1/2 < r < 1$ and CC$[B, B']$. Then by (1), $B \leq \neg B'$. Since (*at least r of*) is right increasing, we have $\|(\textit{at least } r \textit{ of})(A)(\neg B')\| = 1$, which is equivalent to $\|(\textit{at most } 1-r \textit{ of})(A)(B')\| = 1$. Since $1/2 < r < 1$, this entails $\|(\textit{less than } r \textit{ of})(A)(B')\| = 1$, which is equivalent to $\|(\textit{at least } r \textit{ of})(A)(B')\| = 0$. We have thus shown that CC$[B, B'] \Rightarrow$ CC$[(\textit{at least } r \textit{ of})(A)(B), (\textit{at least } r \textit{ of})(A)(B')]$, i.e. (*at least r of*) $\in$ CC→CC+ for $1/2 < r < 1$. The fact that (*more than r of*) $\in$ CC→CC+ for $1/2 \leq r < 1$ can be proved similarly.

To prove that (*exactly r of*) $\in$ −CC→CC for $0 \leq r \leq 1$, we devise a method for constructing counterexample models. Choose natural numbers x and y such that $x/y = r$. Construct two sets A and A' such that $|A| = |A'| = y$ and $A \cap A' = \emptyset$. Choose a subset X of A and a subset X' of A' such that $|X| = |X'| = x$. Then set $U = A \cup A'$ and $B = X \cup X'$. It is easy to check that under this model,

[5] Symmetry and contrapositivity are as defined in Zuber (2006).

[6] In what follows, we assume that the truth condition of a proportional determiner involves $|A|$ in the denominator, e.g. $\|(\textit{at least } r \textit{ of})(A)(B)\| = 1 \Leftrightarrow |A \cap B|/|A| \geq r$. Thus when $A = \emptyset$, these proportional determiners have no truth values.

we have $\mathrm{CC}[A, A']$ and $\|(\textit{exactly r of})(A)(B)\| = \|(\textit{exactly r of})(A')(B)\| = 1$. In other words, we do not have $\mathrm{CC}[(\textit{exactly r of})(A)(B), (\textit{exactly r of})(A')(B)]$, thus completing the proof. □

Based on the above propositions, we can now determine the OPs of some commonly used determiners. For example, let $1/2 < r < 1$ and $A \neq \emptyset$, then since $(\textit{exactly r of}) \leq (\textit{at least r of})$, by Proposition 4 and Proposition 2(a), we immediately have $(\textit{exactly r of}), (\textit{at least r of}) \in -\mathrm{CC}{\to}\mathrm{CC}+$. Similarly, since $(\textit{exactly r of}) \leq (\textit{more than } r - \varepsilon \textit{ of})$ where ε represents an infinitesimal quantity such that $1/2 \leq r - \varepsilon < 1$, by Proposition 4 and Proposition 2(a) again, we have $(\textit{more than } r - \varepsilon \textit{ of}) \in -\mathrm{CC}{\to}\mathrm{CC}+$. Replacing the arbitrary variable $r - \varepsilon$ by r, we can rewrite the last result as $(\textit{more than r of}) \in -\mathrm{CC}{\to}\mathrm{CC}+$ for $1/2 \leq r < 1$. From the above, we can derive even more results.

Since the outer negation of "$(\textit{at least r of})$" is "$(\textit{less than r of})$", by Proposition 1, we have $(\textit{less than r of}) \in -\mathrm{CC}{\to}\mathrm{SC}+$ for $1/2 < r < 1$. Moreover, since the (right) inner negation and (right) dual of "$(\textit{at least r of})$" are "$(\textit{at most } 1-r \textit{ of})$" and "$(\textit{more than } 1-r \textit{ of})$", respectively, by Proposition 1 again, we have $(\textit{at most } 1-r \textit{ of}) \in -\mathrm{SC}{\to}\mathrm{CC}+$ and $(\textit{more than } 1-r \textit{ of}) \in -\mathrm{SC}{\to}\mathrm{SC}+$ for $0 < 1-r < 1/2$. Replacing the arbitrary variable $1-r$ by r, we can rewrite the last results as $(\textit{at most r of}) \in -\mathrm{SC}{\to}\mathrm{CC}+$ and $(\textit{more than r of}) \in -\mathrm{SC}{\to}\mathrm{SC}+$ for $0 < r < 1/2$.

A similar analysis for "$(\textit{more than r of})$" yields the following results: $(\textit{at most r of}) \in -\mathrm{CC}{\to}\mathrm{SC}+$ for $1/2 \leq r < 1$; $(\textit{less than r of}) \in -\mathrm{SC}{\to}\mathrm{CC}+$ and $(\textit{at least r of}) \in -\mathrm{SC}{\to}\mathrm{SC}+$ for $0 < r \leq 1/2$.

For "$(\textit{exactly r of})$", its (right) inner negation may take two forms: "$(\textit{all except r of})$" and "$(\textit{exactly } 1-r \textit{ of})$". Thus, by Proposition 1, we have $(\textit{all except r of}) \in -\mathrm{SC}{\to}\mathrm{CC}+$ for $1/2 < r < 1$ and $(\textit{exactly } 1-r \textit{ of}) \in -\mathrm{SC}{\to}\mathrm{CC}+$ for $0 < 1-r < 1/2$. Replacing the arbitrary variable $1-r$ by r, we can rewrite the last result as $(\textit{exactly r of}) \in -\mathrm{SC}{\to}\mathrm{CC}+$ for $0 < r < 1/2$. Based on the last result and using the fact that the (right) inner negation of "$(\textit{exactly r of})$" is "$(\textit{all except r of})$" and Proposition 1 again, we obtain $(\textit{all except r of}) \in -\mathrm{CC}{\to}\mathrm{CC}+$ for $0 < r < 1/2$.

We next consider the classical determiner "*some*". We have the relation: $(\textit{at least r of}) \leq \textit{some}$ for $0 < r \leq 1/2$ [7], on condition that $A \neq \emptyset$. By virtue of a previous result and Proposition 2(b), we know that $\textit{some} \in \mathrm{SC}{\to}\mathrm{SC}+$ on condition that $A \neq \emptyset$. Note that this condition is essential because when $A = \emptyset$, $\|\textit{some}(\emptyset)(B)\| = 0$ for any B, and so we can never have $\mathrm{SC}[B, B'] \Rightarrow \neg\textit{some}(\emptyset)(B) \leq \textit{some}(\emptyset)(B')$. In other words, $\textit{some} \notin \mathrm{SC}{\to}\mathrm{SC}+$ when $A = \emptyset$. As for the left argument of "*some*", by the symmetry of "*some*" and Proposition 3(a), we know that $\textit{some} \in +\mathrm{SC}{\to}\mathrm{SC}$ subject to certain condition. One can easily find that this condition is $B \neq \emptyset$. The above facts will be represented

[7] Although we also have $(\textit{at least r of}) \leq \textit{some}$ for $1/2 < r < 1$, $(\textit{at least r of}) \in \mathrm{CC}{\to}\mathrm{CC}+$ in this range and this fact cannot be used to derive the OP of "*some*" by virtue of Proposition 2.

succinctly by: $some \in$ +SC→SC+ $(B \neq \emptyset; A \neq \emptyset)$[8]. By using a similar line of reasoning and the fact that "*no*" is the outer negation of "*some*" and is a symmetric determiner, we can find that $no \in$ +SC→CC+ $(B \neq \emptyset; A \neq \emptyset)$.

We next consider "*every*". Since "*every*" is the (right) dual of "*some*", by Proposition 1, we know that $every \in$ CC→CC+ subject to certain condition. This condition is $A \neq \emptyset$ because when $A = \emptyset$, $\|every(\emptyset)(B)\| = 1$ for any B, and so we can never have $\mathrm{CC}[B, B'] \Rightarrow every(\emptyset)(B) \leq \neg every(\emptyset)(B')$. As for the left argument of "*every*", since "*every*" is contrapositive according to Zuber (2006), by Proposition 3(b), we know that $every \in$ +SC→CC subject to certain condition. This condition is $B \neq U$ because when $B = U$, $\|every(A)(U)\| = 1$ for any A, and so we can never have $\mathrm{SC}[A, A'] \Rightarrow every(A)(U) \leq \neg every(A')(U)$. The above facts will be represented succinctly by: $every \in$ +SC→CC ∩ CC→CC+ $(B \neq U; A \neq \emptyset)$. By using a similar line of reasoning and the fact that "(*not every*)" is the outer negation of "*every*" and is a contrapositive determiner, we can find that (*not every*) $\in$ +SC→SC ∩ CC→SC+ $(B \neq U; A \neq \emptyset)$.

Note that in the above, we have only considered the case that a determiner possesses a single OP in its left / right argument. It is logically possible that a determiner may possess more than one OP in one or both of its arguments. However, to explore this possibility we would need more sophisticated notions and so this will be left for future studies.

Based on the above results, we can now derive valid inferential relations between quantified statements. For example, the following are instances exemplifying the facts $every \in$ CC→CC+ and $some \in$ SC→SC+ on condition that $A \neq \emptyset$ (given that CC[TEENAGER, ELDERLY], SC[AGED-OVER-50, AGED-BELOW-51]):

Example 1: (Condition: There is some member in this club.) CC[every member of this club is a teenager, every member of this club is an elderly]

Example 2: (Condition: There is some member in this club.) SC[some member of this club is aged over 50, some member of this club is aged below 51]

4 OPs of Iterated Quantifiers

An adequate theory on opposition inferences should achieve the following. Given an iterated quantifier composed of n constituent determiners[9] in the form:

$$Q_1(A_1)(\{x_1 : \ldots Q_n(A_n)(\{x_n : B(x_1, \ldots x_n)\}) \ldots\}) \tag{7}$$

[8] The conditions $B \neq \emptyset; A \neq \emptyset$ are ordered such that the first (second) condition corresponds to the first (second) occurrence of "+" in +SC→SC+.

[9] Iterated quantifiers refer to polyadic quantifiers constructed from monadic quantifiers by "iteration" (Peters and Westerståhl (2006)). In this paper we only consider iterated quantifiers constructed from determiners.

we hope to determine the OPs of this iterated quantifier in the predicates $A_1, \dots A_n, B$ based on the OPs of $Q_1, \dots Q_n$. To this end, we first define the notion of "OP-chain":

Definition 3: Let X be a predicate under an iterated quantifier. Suppose X is within the i_k argument of $Q_k (1 \leq k \leq n)$, i_{k-1} argument of $Q_{k-1}, \dots i_1$ argument of Q_1, where each of $i_k, i_{k-1}, \dots i_1$ is one of {left, right} and $Q_k, Q_{k-1}, \dots Q_1$ are constituent determiners of the iterated quantifier ordered from the innermost to the outermost layers. Then X has an OP-chain $\langle R_k, R_{k-1}, \dots R_0 \rangle$, where each of $R_k, R_{k-1}, \dots R_0$ is one of {CC, SC}, iff Q_k is $R_k \to R_{k-1}$ in the i_k argument, Q_{k-1} is $R_{k-1} \to R_{k-2}$ in the i_{k-1} argument, $\dots Q_1$ is $R_1 \to R_0$ in the i_1 argument.

For instance, in the following iterated quantifier:

$$(\textit{at most 1/2 of})(A_1)(\{x_1 : \textit{no}(A_2)(\{x_2 : B(x_1, x_2)\})\}) \tag{8}$$

A_2 is within the left argument of "*no*" and right argument of "(*at most 1/2 of*)". Since *no* $\in$ +SC→CC on condition that its right argument is non-empty and (*at most 1/2 of*) $\in$ CC→SC+ on condition that its left argument is non-empty, A_2 has an OP-chain ⟨SC, CC, SC⟩ on condition that $A_1 \neq \emptyset \wedge \{x_2 : B(x_1, x_2)\} \neq \emptyset$. Similarly, one can easily check that B has an OP-chain ⟨SC, CC, SC⟩ on condition that $A_1 \neq \emptyset \wedge A_2 \neq \emptyset$, while A_1 has no OP-chain.

To facilitate the discussion below, we first state a proposition.

Proposition 5: Let $P(x_1, \dots x_n)$ and $P'(x_1, \dots x_n)$ be n-ary predicates and R be one of {CC, SC}, then $\mathrm{R}[P, P'] \Rightarrow \mathrm{R}[\{x_i : P(y_1, \dots y_{i-1}, x_i, y_{i+1}, \dots y_n)\}, \{x_i : P'(y_1, \dots y_{i-1}, x_i, y_{i+1}, \dots y_n)\}]$ for any $1 \leq i \leq n$ and any particular set of $y_1, \dots y_{i-1}, y_{i+1}, \dots y_n$.

Proof: Here we only prove the case in which R = CC. The case in which R = SC is similar. Suppose $\mathrm{CC}[P, P']$. By (1), this is equivalent to $P \leq \neg P'$. Then for any particular set of $y_1, \dots y_{i-1}, y_{i+1}, \dots y_n$ and any arbitrary x_i, we have $P(y_1, \dots y_{i-1}, x_i, y_{i+1}, \dots y_n) \leq \neg P'(y_1, \dots y_{i-1}, x_i, y_{i+1}, \dots y_n)$, and so we have $\{x_i : P(y_1, \dots y_{i-1}, x_i, y_{i+1}, \dots y_n)\} \leq \{x_i : \neg P'(y_1, \dots y_{i-1}, x_i, y_{i+1}, \dots y_n)\}$. But $\{x_i : \neg P'(y_1, \dots y_{i-1}, x_i, y_{i+1}, \dots y_n)\}$ can be rewritten as $\neg\{x_i : P'(y_1, \dots y_{i-1}, x_i, y_{i+1}, \dots y_n)\}$. Thus, by (1) again, we have $\mathrm{CC}[\{x_i : P(y_1, \dots y_{i-1}, x_i, y_{i+1}, \dots y_n)\}, \{x_i : P'(y_1, \dots y_{i-1}, x_i, y_{i+1}, \dots y_n)\}]$. □

We can now deduce a condition for determining the OPs of the iterated quantifier (7) in its predicates based on the OPs of its constituent determiners. We focus on the predicate B (the other predicates can be similarly treated). Let B have an OP-chain $\langle R_n, R_{n-1}, R_{n-2}, \dots R_0 \rangle$ and $R_n[B, B']$. By Proposition 5, we have $R_n[\{x_n : B(x_1, \dots x_n)\}, \{x_n : B'(x_1, \dots x_n)\}]$ for any $x_1, \dots x_{n-1}$. Moreover, by the definition of OP-chain, Q_n is $R_n \to R_{n-1}$ in the argument $\{x_n : B(x_1, \dots x_n)\}$, and so we have $R_{n-1}[Q_n(A_n)(\{x_n : B(x_1, \dots x_n)\}), Q_n(A_n)(\{x_n :$

$B'(x_1, \ldots x_n)\})]$. The above reasoning can be seen as a kind of "upward derivation": from the R_n relation at the B-level, we derive the R_{n-1} relation at the Q_n-level. Now $Q_n(A_n)(\{x_n : B(x_1, \ldots x_n)\})$ can be seen as an $(n-1)$-ary predicate (with $x_1, \ldots x_{n-1}$ as arguments). Thus, we can carry out the aforesaid upward-derivation again and derive a R_{n-2} relation at the Q_{n-1}-level. The process of determining the OPs of the iterated quantifier (7) in B is essentially a repetition of this upward derivation. After n rounds of derivation, we will finally derive the R_0 relation at the Q_1 level. The net effect is thus $R_n(B, B') \Rightarrow R_0[Q_1(A_1)(\{x_1 : \ldots Q_n(A_n)(\{x_n : B(x_1, \ldots x_n)\}) \ldots\}), Q_1(A_1)(\{x_1 : \ldots Q_n(A_n)(\{x_n : B'(x_1, \ldots x_n)\}) \ldots\})]$, showing that the iterated quantifier is $R_n \to R_0$ in B.

The above derivation relies on the fact that B has an OP-chain. This may not be the case either when at least one of $Q_1, \ldots Q_n$ possesses none of the OPs, or when the OPs possessed by $Q_1, \ldots Q_n$ do not form a chain. In either case, the absence of the OP-chain blocks the upward derivation. Based on the above discussion, we can thus formulate the following condition: let X be a predicate under an iterated quantifier Q,

$$Q \text{ is } R_k \to R_0 \text{ in } X \text{ iff } X \text{ has an OP-chain } \langle R_k, \ldots R_0 \rangle \tag{9}$$

We now use (9) to determine the OPs of (8) in its predicates. Previously we have already found that A_2 and B both have the OP-chain $\langle$SC, CC, SC$\rangle$ subject to different conditions, whereas A_1 has no OP-chain. Thus, according to (9), we know that (8) is SC→SC in A_2 on condition that $A_1 \neq \emptyset \wedge \{x_2 : B(x_1, x_2)\} \neq \emptyset$ and SC→SC in B on condition that $A_1 \neq \emptyset \wedge A_2 \neq \emptyset$. Moreover, (8) possesses none of the OPs in A_1. From the above result, we can derive the following (by letting A_1 = CLUB, A_2 = AGED-OVER-50, A'_2 = AGED-BELOW-51, B = ADMIT-AS-MEMBERS):

Example 3: (Condition: There is at least a club and every club admits somebody as members.) SC[at most 1/2 of the clubs admit nobody aged over 50 as members, at most 1/2 of the clubs admit nobody aged below 51 as members.]

The derivation process of (9) is not exclusively valid for (7). In fact the condition in (9) can also be applied to iterated quantifiers in a form different than (7). Consider the following:

$$no(A \cap \{x : some(B)(\{y : C(x, y)\})\})(D) \tag{10}$$

The above iterated quantifier represents a quantified statement whose subject contains a relative clause which is another quantified statement. Let's determine the OP of (10) in the predicate B by using (9). Since B falls within the left arguments of "*some*" and "*no*", which are +SC→SC and +SC→CC, respectively, both on condition that their right arguments are non-empty, B has an OP-chain $\langle$SC, SC, CC$\rangle$. By (9), (10) is SC→CC in B subject to the condition that $\{y : C(x, y)\} \neq \emptyset \wedge D \neq \emptyset$. From the above result, we can derive the following (by letting A = COMPANY, B = AGED-OVER-50, B' = AGED-BELOW-51, C = EMPLOY, D = GO-BANKRUPT):

Example 4: (Condition: Every company employs somebody and some company went bankrupt.) CC[no company employing somebody aged over 50 went bankrupt, no company employing somebody aged below 51 went bankrupt]

Note that monotonicity inferences of iterated quantifiers are governed by the same condition as opposition inferences. We can define an analogous notion of MON-chain by replacing {CC, SC} with $\{\leq, \geq\}$ in Definition 3 and modify the condition in (9) by replacing "OP-chain" with "MON-chain". The modified condition can then be used to determine the monotonicities of iterated quantifiers in its predicates.

For illustration, consider the iterated quantifier in (8) again. Let's determine the monotonicity of (8) in the predicate A_2. Since A_2 is within the left argument of "*no*" and right argument of "(*at most 1/2 of*)", and "*no*" is left decreasing while "(*at most 1/2 of*)" is right decreasing, A_2 has a MON-chain $\langle \leq, \geq, \leq \rangle$ (or equivalently, $\langle \geq, \leq, \geq \rangle$). According to the modified version of condition (9), we know that (8) is $\leq \rightarrow \leq$ (or equivalently $\geq \rightarrow \geq$), i.e. increasing, in A_2. This result is in accord with that obtained by using van Eijck (2007)'s "monotonicity calculus".

5 Concluding Remarks

According to van Eijck (2007), monotonicity inferences are an important type of inferences in modern Natural Logic. Even syllogistics, the most important type of classical inferences, are subsumable under monotonicity inferences. By proposing the study on "inferences with exclusion premises", van Benthem (2008) has opened up a new direction of studies on Natural Logic. This paper is an implementation and generalization of van Benthem (2008)'s proposal and a contribution to the studies on Natural Logic. We have proposed a number of results by which we can determine the OPs of determiners and iterated quantifiers composed of constituent determiners, and derive valid inferential relations between quantified statements.

Nevertheless, one may criticize that the inferential relations derived from the OPs of determiners are too weak. For instance, by (1) the inferential relation in Example 1 above can be rewritten as the following entailment:

> Every member of this club is a teenager. ⇒ Not every member of this club is an elderly.

Although valid, the conclusion above seems too weak because intuitively, one would expect that the proper conclusion of the above inference should be "No member of this club is an elderly".

However, entailment is not the only type of inferential relations. In some situations, we do need to establish some other types of inferential relations (such as the CC or SC relation) between sets / propositions. These situations do not only include solving logical puzzles, but also include linguistic uses. One such use is to determine the incompatibility between two sets / propositions. For instance, from the fact that *every* $\in -\text{CC}\rightarrow\text{CC}+$[10], we know that "clubs every member of which is a teenager" and "clubs every member of which is an elderly" are incompatible, whereas "clubs of which every teenager is a member" and "clubs of which every elderly is a member" are not.

The determination of incompatibility can help us differentiate between entailments and implicatures, especially the "alternate-value implicatures" studied by Hirsch-berg (1975). For instance, in the following discourse, B's conclusion is a logical entailment inferred from A's utterance:

A: This is a club every member of which is a teenager.
B: So it is not a club every member of which is an elderly.

whereas in the following discourse, B's conclusion (inferred from A's utterance under suitable context) should be seen as an alternate-value implicature that is cancellable:

A: This is a club of which every teenager is a member.
B: So it is not a club of which every elderly is a member.

Note that the difference between the aforesaid two discourses is analogous to the difference between the following two discourses:

A: She is my enemy. B: So she is not your platonic friend.
A: She is my colleague. B: So she is not your platonic friend.

References

1. van Benthem, J.: A Brief History of Natural Logic. Technical Report PP-2008-05, Institute for Logic, Language and Computation (2008)
2. van Eijck, J.: Natural Logic for Natural Language. In: ten Cate, B.D. and Zeevat, H.W. (eds.), Logic, Language, and Computation; 6th International Tbilisi Symposium on Logic, Language, and Computation. Springer-Verlag, Berlin, 216-230 (2007)
3. Hirschberg, J.B.: A Theory of Scalar Implicature, PhD thesis, University of Pennsylvania (1985)
4. Icard, T.: Exclusion and Containment in Natural Language. Accepted to Studia Logica (2011)
5. MacCartney, B.: Natural Language Inference. PhD thesis, Stanford University (2009)
6. MacCartney, B. and Manning, C.D.: An extended model of natural logic. In: Proceedings of the Eighth International Conference on Computational Semantics (2009)
7. Peters, S. and Westerståhl, D.: Quantifiers in Language and Logic. Clarendon Press, Oxford (2006)
8. Zuber, R.: Symmetric and contrapositional quantifiers. Journal of Logic, Language and Information, Vol. 16, No 1, 1-13 (2006)

[10] Note that in Section 3, we have only established that *every* $\in \text{CC}\rightarrow\text{CC}+$. But since *every* = (*exactly 100% of*), by Proposition 4, we know that *every* $\in -\text{CC}\rightarrow\text{CC}$.