A Semantic Model for Vague Quantifiers combining Fuzzy Theory and Supervaluation Theory*

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Abstract. This paper introduces a semantic model for vague quantifiers (VQs) combining Fuzzy Theory (FT) and Supervaluation Theory (ST), which are the two main theories on vagueness, a common source of uncertainty in natural language. After comparing FT and ST, I will develop the desired model and a numerical method for evaluating truth values of vague quantified statements, called the Modified Glöckner's Method, that combines the merits and overcomes the demerits of the two theories. I will also show how the model can be applied to evaluate truth values of complex quantified statements with iterated VQs.

Keywords. vague quantifiers, Generalized Quantifier Theory, Fuzzy Theory, Supervaluation Theory, Modified Glöckner's Method

1 Introduction

Vagueness is a common source of uncertainty in natural language. No doubt vague quantifiers (VQs) constitute an important type of quantifiers, the target of study of the Generalized Quantifier Theory (GQT). However, since it is difficult to model vagueness under standard Set Theory, the study on VQs has remained a weak point of GQT.

In GQT, the most typical approach of representing the truth condition of a VQ is to represent it as a comparison between an expression consisting of the VQ's arguments and a context-dependent standard. For example, according to [11], there are three interpretations of "*many*". The truth condition of "$many^2$" is as follows:

$$many^2(A)(B) \leftrightarrow |A \cap B| \geq k|A| \quad (1)$$

where $k \in (0, 1)$ is a context-dependent constant. This condition says that "Many As are B" is true whenever the proportion of those As that are B among all As is at least

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as great as a standard, i.e. k, representing the threshold of "many". Since k is dependent on context, the above condition may yield different truth values for two different quantified statements "Many A_1s are B_1" and "Many A_2s are B_2" even if $|A_1| = |A_2|$ and $|A_1 \cap B_1| = |A_2 \cap B_2|$.

While this approach is most straightforward, what it genuinely reflects is the context dependence rather than the vagueness of VQs. In this paper, I will leave aside the issue of context dependence and concentrate on the vagueness of VQs. Moreover, I will only deal with VQs in a general manner and will not work out the detailed semantics of any particular VQ. Since vague concepts are characterized by blurred boundaries and uncertain membership, we need to invoke theories that deal with such phenomena. In the next section, I will introduce two such theories: Fuzzy Theory and Supervaluation Theory. The former is further divided into two approaches: the Fuzzy Set Cardinality Approach and the Quantifier Fuzzification Mechanism Approach.

2 Basic Theories for VQs

2.1 Fuzzy Theory (Fuzzy Set Cardinality Approach)

Fuzzy Theory (FT) is a cover term for all those theories that are based on or derived from the Fuzzy Set Theory developed by [14]. Ever since [14], FT has become a new paradigm and is widely applied in many areas. Under FT, vague concepts are modeled by fuzzy sets, which differ from crisp sets (i.e. non-fuzzy sets) in one important aspect: instead of having sharp boundaries between members and non-members, every individual in the universe belongs to a fuzzy set to a certain degree ranging from absolute membership to absolute non-membership. By using $\|p\|$ to denote the truth value of a proposition p, we can represent this degree by a membership degree function (MDF), $\|x \in S\|$, which outputs a numerical value in [0, 1] representing the degree to which an individual x belongs to a fuzzy set S[1]. For example, $\|j \in TALL\|$ = 0.7 means that John is tall to the degree 0.7. Sometimes, the MDF may take the form of a mathematical function that depends on a numerical value (henceforth called the "input" of the MDF). For example, as the tallness of a person depends on the person's height, the aforesaid MDF for TALL may take the alternative form $\|h \in TALL\|$, where h represents the height of a person.

Fuzzy theorists have also defined certain crisp sets corresponding to each fuzzy set. Let X be a fuzzy set and α be a real number in [0, 1]. Then the α-cut (denoted $X_{\geq\alpha}$), and strict α-cut (denoted $X_{>\alpha}$) of X are defined as follows (in what follows, U represents the universe):

$$X_{\geq\alpha} = \{x \in U: \|x \in X\| \geq \alpha\} \quad (2)$$

$$X_{>\alpha} = \{x \in U: \|x \in X\| > \alpha\} \quad (3)$$

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1 In the literature, the MDF is often expressed as $\mu_S(x)$. In this paper I use $\|x \in S\|$ instead for convenience.

Another characteristic of FT is that it treats Boolean operators (BOs) as truth functions such as[2]:

$$\| p \wedge q \| = \min(\{ \| p \|, \| q \| \}) \tag{4}$$

$$\| p \vee q \| = \max(\{ \| p \|, \| q \| \}) \tag{5}$$

$$\| \neg p \| = 1 - \| p \| \tag{6}$$

Inspired by GQT, fuzzy theorists also tried to formalize theories about VQs, with [15] and [12-13] being the earlier attempts. Since VQs can be seen as fuzzy sets of numbers, they can also be modeled by MDFs. For example, borrowing ideas from [1], we may represent the VQ "(*about 10*)" by the following MDF:

$$\| (\textit{about 10})(A)(B) \| = T_{-4, -1, 1, 4}(|A \cap B| - 10) \tag{7}$$

There are several points to note concerning the above formula. First, I have adopted [8]'s notation that represents a quantified statement in the form of a tripartite structure "Q(A)(B)" where Q, A and B represent the quantifier and its two arguments, respectively[3]. Syntactically, these two arguments correspond to the subject (excluding the quantifier) and the predicate of the quantified statement.

Second, the above formula makes use of a piecewise-defined function $T_{a, b, c, d}(x)$ with the following definition:

$$T_{a, b, c, d}(x) = \begin{cases} 0, & \text{if } x < a \\ (x - a) / (b - a), & \text{if } a \le x < b \\ 1 & \text{if } b \le x \le c \\ (d - x) / (d - c), & \text{if } c < x \le d \\ 0, & \text{if } x > d \end{cases} \tag{8}$$

The above function is named "T", standing for "trapezoid", because its graph has a trapezoidal shape. Figure 1 shows the graph of $T_{-4, -1, 1, 4}$:

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[2] In the literature, there is a whole range of possible definitions of BOs. What follows are the "standard" definitions of the most commonly used BOs.

[3] In this paper, I only consider VQs that have two arguments. Using standard GQT notation, such kind of VQs belongs to type <1,1> quantifiers, also called "determiners".

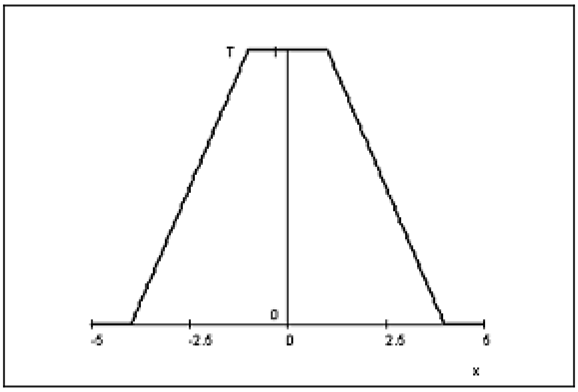

Fig. 1. $T_{-4, -1, 1, 4}$

When the parameters are such that a = b or c = d, since there is no x such that $a \leq x < a$ or $c \leq x < c$, the 2nd or 4th piece of (8) would disappear. In these cases, T becomes degenerate and its graph is shaped like half of a trapezoid. For example, Figure 2 shows the graph of $T_{-0.25, -0.1, \infty, \infty}$:

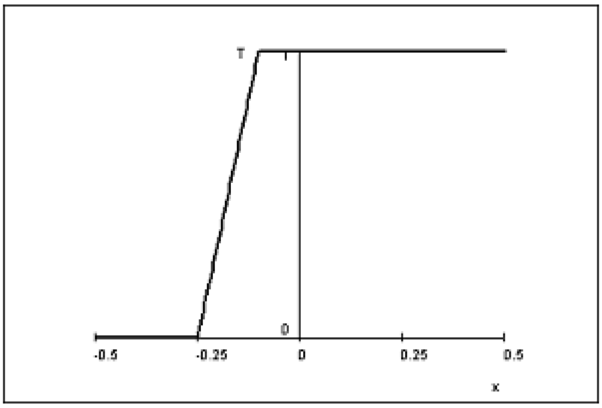

Fig. 2. $T_{-0.25, -0.1, \infty, \infty}$

Note that the function T given in (8) gives just one possible example of MDFs that may be used for representing VQs. In fact, any function whose general shape is similar to T can also serve the same purpose. More specifically, this function should be a function whose domain can be partitioned into 5 parts such that the values at the 1st, 2nd, 3rd, 4th and 5th parts are constantly 0, increasing, constantly 1, decreasing and constantly 0, respectively[4].

Using the MDFs for VQs, one can then evaluate the truth values of sentences containing VQs. However, the evaluation of truth values of these sentences

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[4] In case the function becomes degenerate, then some of the aforesaid parts would disappear.

sometimes may involve some complications. For example, consider the following sentence:

$$\text{About 10 tall girls sang.} \tag{9}$$

This sentence contains the VQ "(*about 10*)". According to (7), the input of the MDF for "(*about 10*)(TALL-GIRL)(SING)" is the number |TALL-GIRL ∩ SING| – 10. However, since TALL-GIRL ∩ SING is fuzzy, its cardinality is not well defined. We now encounter the following problem: how can we evaluate the truth value of (9) if we cannot say for sure how many "tall girls" there are?

The solution of the early fuzzy theorists is to generalize the notion of crisp set cardinality to fuzzy set cardinality, which may have different definitions. One definition (called the Sigma Count) is the sum of the membership degrees of all individuals in the universe with respect to the fuzzy set. For example, if the fuzzy set TALL-GIRL ∩ SING = {1/a, 0.7/b, 0.5/c, 0.2/d, 0.1/e}[5], then the Sigma Count of this set is 1 + 0.7 + 0.5 + 0.2 + 0.1 = 2.5. Using this cardinality, the truth value of (9) is then equal to $\| 2.5 \in (\textit{about 10}) \|$, which is equal to 0 according to (7). This shows that (9) is absolutely false with respect to the aforesaid fuzzy set TALL-GIRL ∩ SING. This is in accord with our intuition because according to that fuzzy set, there are only 2 members (i.e. a and b) who may be counted as singing tall girls with a relatively high certainty, and 2 absolutely falls short of being "about 10".

2.2 Fuzzy Theory (Quantifier Fuzzification Mechanism Approach)

Later, some scholars (e.g. [1], [4-5], [10]) realized the demerits of the old approach, which was able to treat only certain types of VQs and could not be applied to more general types of VQs. Moreover, since different notions of fuzzy set cardinality were used for different VQs, there was not a uniform treatment for various types of VQs.

Instead of using the concept of fuzzy set cardinality, they proposed the concept of quantifier fuzzification mechanisms (QFMs). This approach distinguishes two types of VQs: semi-fuzzy and fuzzy quantifiers. Semi-fuzzy quantifiers are those VQs that only take crisp sets as arguments; while fuzzy quantifiers are those VQs that may take either crisp or fuzzy sets as arguments. Note that the distinction between semi-fuzzy and fuzzy quantifiers has nothing to do with the meaning of the VQs. Thus, the same linguistic quantifier such as "(*about 10*)" may manifest either as a semi-fuzzy or a fuzzy quantifier, depending on the types of its arguments[6].

Under this approach, all VQs are initially modeled as semi-fuzzy quantifiers. This has the advantage of greatly simplifying the semantics of VQs. We only need to

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5 Here I adopt a notation used by fuzzy theorists under which a fuzzy set S is represented in the form $\{r_1/x_1, r_2/x_2, \ldots\}$ where x_is are individuals and r_is are their respective membership degrees, i.e. $r_i = \| x_i \in S \|$. In case the membership degree of an individual is 0, it is not listed.

6 Since crispness can be seen as a special case of fuzziness, any crisp quantifier such as "*every*" can be seen as a semi-fuzzy or fuzzy quantifier, depending on the types of its arguments.

formulate an appropriate MDF or truth condition for each VQ without worrying about its inputs because all inputs are crisp. The evaluation of truth values of sentences involving semi-fuzzy quantifiers is easy: we only need to plug the crisp inputs into the appropriate MDFs or truth conditions. When it comes to a sentence involving fuzzy quantifiers with fuzzy inputs (such as (9)), we have to make use of a QFM, which is in fact a mapping that transforms a semi-fuzzy quantifier to a fuzzy quantifier.

Among the QFM approach, [4]'s framework has certain merits compared with its competitors in that it proposes a number of axioms that an adequate QFM should satisfy[7]. These axioms guarantee that the QFM will preserve crisp arguments, the identity truth function and monotonicities of a VQ as well as its arguments, and that the QFM will commute with the operations of argument transposition, argument insertion, external negation, internal negation, internal meet (as well as other Boolean) operators and functional application. Note that the aforesaid properties / operations are crucial to the study of quantifiers under GQT.

Next I introduce a QFM proposed in [4][8]. First let X be a fuzzy set and γ be a real number in [0, 1] which is called the "cut level". We can reduce X into two crisp sets X_γ^{min} and X_γ^{max} at the cut level γ using the following formulae: for $\gamma > 0$ [9],

$$X_\gamma^{min} = X_{\geq 0.5 + 0.5\gamma}; \quad X_\gamma^{max} = X_{> 0.5 - 0.5\gamma} \tag{10}$$

Based on the above, we can then define a family of crisp sets associated with X:

$$T_\gamma(X) = \{Y: X_\gamma^{min} \subseteq Y \subseteq X_\gamma^{max}\} \tag{11}$$

Then let Q be a semi-fuzzy quantifier and $X_1, \dots X_n$ be n fuzzy sets. Now for each of $X_1, \dots X_n$ we can define $T_\gamma(X_1), \dots T_\gamma(X_n)$. For each possible combination of $Y_1 \in T_\gamma(X_1), \dots Y_n \in T_\gamma(X_n)$, we can evaluate $\| Q(Y_1, \dots Y_n) \|$ by using a suitable MDF or truth condition because $Y_1, \dots Y_n$ are crisp sets. Then we aggregate the various values of $\| Q(Y_1, \dots Y_n) \|$ for all possible combinations of $Y_1, \dots Y_n$ into $\| Q_\gamma(X_1, \dots X_n) \|$ [10] by the following formula:

$$\| Q_\gamma(X_1, \dots X_n) \| = m_{0.5}(\{ \| Q(Y_1, \dots Y_n) \| : Y_1 \in T_\gamma(X_1), \dots Y_n \in T_\gamma(X_n)\}) \tag{12}$$

where $m_{0.5}$, called the "generalized fuzzy median", is defined as follows. Let Z be a set of real numbers, then

$$\inf(Z), \quad \text{if } |Z| \geq 2 \wedge \inf(Z) > 0.5 \tag{13}$$

[7] Actually, Glöckner used the term "determiner fuzzification schemes" (DFSs) in [4]. In [5], he used QFMs as a general term for all mappings that map a semi-fuzzy quantifier to a fuzzy quantifier and used DFSs to refer to those QFMs that satisfy his axioms. To simplify notation, in what follows I will just use the umbrella term QFM.

[8] Glöckner has proposed a number of QFMs that satisfy all his axioms. This paper only discusses the simplest one.

[9] There are in fact separate definitions for X_γ^{min} and X_γ^{max} at $\gamma = 0$. But since a single point will only contribute the value 0 to a definite integral to be introduced below, we do not need to consider the case $\gamma = 0$ for computational purpose.

[10] Note that here Q_γ should be seen as a fuzzy quantifier evaluated at the cut level γ.

$$m_{0.5}(Z) = \begin{cases} \sup(Z), & \text{if } |Z| \geq 2 \wedge \sup(Z) < 0.5 \\ 0.5, & \text{if } (|Z| \geq 2 \wedge \inf(Z) \leq 0.5 \wedge \sup(Z) \geq 0.5) \vee (Z = \emptyset) \\ r, & \text{if } Z = \{r\} \end{cases}$$

Now for each cut level γ, we have a corresponding value $\| Q_\gamma(X_1, \ldots X_n) \|$. Finally we need to combine all these values into one value. According to [4], there are various methods of combination, one such method (which is denoted by "M" in [4]) is to use the standard definite integral[11]:

$$\| M(Q)(X_1, \ldots X_n) \| = \int_0^1 \| Q_\gamma(X_1, \ldots X_n) \| d\gamma \qquad (14)$$

Although the above formula appears as an integral, in practical calculation of linguistic applications involving finite universes, we often only need to consider a finite number of variations of γ and $\| Q_\gamma(X_1, \ldots X_n) \|$ is constant at each such γ, and so the integral above often reduces to a sum, which can be seen as a "weighted average" of $\| Q_\gamma(X_1, \ldots X_n) \|$ at the various γs.

2.3 Supervaluation Theory

The Supervaluation Theory (ST) for vagueness is a keen competitor of the FT. Some supervaluation theorists, such as [3] and [6-7], pointed out certain flaws of FT. The most serious one is that FT cannot correctly predict the truth values of certain statements that must be true / false by virtue of traditional logical laws or intuition with respect to a model (such statements are called "penumbral connections" in [3]).

Consider the following model:

M1 $\qquad U = \{j, m\}; \text{TALL} = \{0.5/j, 0.3/m\}$

Intuitively, according to this model, the truth values of the following sentences should both be absolutely false (where John and Mary are represented by j and m above):

John is tall and John is not tall. (15)

Mary is tall and John is not tall. (16)

But using the truth functions for BOs (4) – (6), the calculation results show that the above sentences are both true to a certain degree under FT:

$$\| (15) \| = \| j \in \text{TALL} \| \wedge \| j \notin \text{TALL} \| = \min(\{0.5, 1 - 0.5\}) = 0.5$$

$$\| (16) \| = \| m \in \text{TALL} \| \wedge \| j \notin \text{TALL} \| = \min(\{0.3, 1 - 0.5\}) = 0.3$$

Supervaluation theorists point out that the above wrong predictions arise from the wrong assumption that BOs are truth functional when applied to vague concepts.

[11] In the following formula, "M" should be seen as a QFM that transforms the semi-fuzzy quantifier Q to a fuzzy quantifier M(Q).

Note that the aforesaid flaw does not hinge on the particular definitions of BOs. It is argued in [7] that the definitions of BOs are subject to various plausible constraints. For example, one may hope that the definitions will preserve $p \rightarrow q \equiv \neg p \vee q$, or that $p \rightarrow p$ is always true. But unfortunately, no set of definitions can satisfy all these plausible constraints under FT.

Supervaluation theorists view vague concepts as truth value gaps and evaluate the truth values of vague sentences by means of complete specifications. A complete specification is an assignment of the truth value 1 or 0 to every individual with respect to the relevant vague sets in a sentence. In other words, a complete specification eliminates the truth value gaps and makes a vague sentence precise. Thus, this process is called "precisification". If a sentence is true (false) on all admissible complete specifications, then we say that it is true (false)[12]. Otherwise, it has no truth value.

The concept of "admissible" is very important in ST. Let's use model M1 to illustrate this point. This model contains two individuals: j and m such that both are borderline cases of the vague set TALL with j taller than m. Here is a list of all admissible complete specifications for M1: (i) $\|j \in \text{TALL}\| = 1$, $\|m \in \text{TALL}\| = 1$; (ii) $\|j \in \text{TALL}\| = 1$, $\|m \in \text{TALL}\| = 0$; (iii) $\|j \in \text{TALL}\| = 0$, $\|m \in \text{TALL}\| = 0$. The above list does not include $\|j \in \text{TALL}\| = 0$, $\|m \in \text{TALL}\| = 1$ because it is inadmissible to assign a person to the set of TALL without at the same time assigning another person who is even taller to TALL.

Having identified the admissible specifications, we can then evaluate $\|(15)\|$ and $\|(16)\|$. Since (15) and (16) are both false on all of (i) – (iii) above, we obtain $\|(15)\| = \|(16)\| = 0$, in conformity with our intuition.

Thus, ST provides an alternative method that can deal with penumbral connections correctly. The same method can also be used to evaluate truth values of sentences containing VQs, although the precisification process may be more complicated. Using (9) as an example, the precisification process will involve two levels. At the first level, the vague concept "tall girl" will be precisified, after which we obtain a set TALL-GIRL ∩ SING whose cardinality is known. Then, at the second level, the VQ "about 10" will be precisified based on the aforesaid cardinality.

The main weakness of ST is that it cannot distinguish different degrees of vagueness because it treats all borderline cases alike as truth value gaps. The evaluation of truth values of vague sentences under ST is uninteresting because all those vague sentences other than penumbral connections have no truth values. Moreover, in applied studies such as Control Theory, Artificial Intelligence, etc., the concept of membership degrees is of great use. That is why while FT has become very popular in applied studies, ST is only popular in theoretical studies.

As a matter of fact, [6] has discussed how to develop a version of ST that incorporates the notion of degrees. More recently, [2] even showed that FT and ST, though often seen to be incompatible with each other, can in fact be combined. In the next section, I will propose such a combined theory.

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12 In [3] the terms "super-true" ("super-false") were used to denote propositions that are true (false) on all admissible complete specifications. To simplify notation, I will just call such propositions "true" ("false").

3 Combining FT and ST

3.1 The Modified Glöckner's Method

Although all borderline cases can be treated as truth value gaps, they may behave differently in the process of precisification. For example, among all admissible complete specifications in which individuals are assigned to the set TALL, a taller person x is more likely to be assigned full membership of TALL than a shorter person y, because whenever y is assigned full membership of TALL in an admissible specification, x must also be so, but not vice versa. An individual x's membership degree with respect to a vague set S may thus be seen as representing the likelihood of x being assigned to S in an admissible specification. By reinterpreting membership degrees in this way, we have established a link between FT and ST and the semantic model for VQs developed below will follow the tradition of FT by using MDFs as a measure of truth values of VQs.

How are we to evaluate the truth values of vague sentences such as (15) and (16)? As mentioned ahove, the traditional FT approach of treating BOs as truth functions like (4) – (6) has to be abandoned. Neither can we use ST's method because we now want to distinguish an infinite number of truth values. Fortunately, Glöckner's method in [4] as introduced in Subsection 2.2 can meet our requirements.

The essence of Glöckner's method in [4] is to reduce a sentence with vague arguments to sentences with crisp arguments at different cut levels. The truth values of these sentences with crisp arguments are then evaluated using the MDFs or truth conditions and aggregated into the truth values of a vague quantified sentence at a cut level. Finally, the truth values at all cut levels are combined into a "weighted average", which is then taken to be the truth value of the original sentence.

Using the aforesaid method, there is no need to invoke (4) – (6). Moreover, the aforesaid reduction process can be seen as a precisification process and the family of crisp sets $T_\gamma(X)$ as defined in (11) can be seen as a set of complete specifications of X. To guarantee that these are also admissible specifications, we need to modify the definition of $T_\gamma(X)$ as shown below:

$$T_\gamma(X) = \{Y: X_\gamma^{min} \subseteq Y \subseteq X_\gamma^{max} \wedge \text{Y represents an admissible complete specification of X}\} \quad (17)$$

Glöckner's method with the above modification will henceforth be called the Modified Glöckner's Method (MGM). With MGM, we can evaluate $\|(15)\|$ and $\|(16)\|$ with respect to M1. Since the result of $\|(15)\|$ is obvious, I only show the evaluation of $\|(16)\|$. In order to use MGM, we first need to express (16) as a conjoined quantified statement. One way is to make use of the quantifier "*every*" satisfying the truth condition $every(A)(B) \leftrightarrow A \subseteq B$:

$$every(\{m\})(TALL) \wedge \neg every(\{j\})(TALL) \quad (18)$$

Now, for $0 < \gamma \leq 0.4$, we have by (10), $TALL_\gamma^{min} = \varnothing$, $TALL_\gamma^{max} = \{j\}$, By (17), $T_\gamma(TALL) = \{\varnothing, \{j\}\}$ since both $\varnothing$ and $\{j\}$ represent admissible complete specifications. Then, we have

$$\|every_\gamma(\{m\})(TALL) \wedge \neg every_\gamma(\{j\})(TALL)\|$$

$$= m_{0.5}(\{\|every(\{m\})(Y) \wedge \neg every(\{j\})(Y)\| : Y \in T_\gamma(TALL)\}) \quad \text{by (12)}$$

$$= m_{0.5}(\{\|every(\{m\})(\varnothing) \wedge \neg every(\{j\})(\varnothing)\|, \|every(\{m\})(\{j\}) \wedge \neg every(\{j\})(\{j\})\|\})$$

$$= m_{0.5}(\{0\})$$

$$= 0 \quad \text{by (13)}$$

For $0.4 < \gamma \leq 1$, $TALL_\gamma^{min} = \varnothing$, $TALL_\gamma^{max} = \{j, m\}$. By (17), $T_\gamma(TALL) = \{\varnothing, \{j\}, \{j, m\}\}$ since $\varnothing$, $\{j\}$ and $\{j, m\}$ represent admissible complete specifications. Note that although $\varnothing \subseteq \{m\} \subseteq \{j, m\}$, $\{m\}$ is not included in $T_\gamma(TALL)$ because $\{m\}$ represents the inadmissible complete specification $\|j \in TALL\| = 0$, $\|m \in TALL\| = 1$. Then, we have

$$\|every_\gamma(\{m\})(TALL) \wedge \neg every_\gamma(\{j\})(TALL)\|$$

$$= m_{0.5}(\{\|every(\{m\})(Y) \wedge \neg every(\{j\})(Y)\| : Y \in T_\gamma(TALL)\}) \quad \text{by (12)}$$

$$= m_{0.5}(\{\|every(\{m\})(\varnothing) \wedge \neg every(\{j\})(\varnothing)\|, \|every(\{m\})(\{j\}) \wedge \neg every(\{j\})(\{j\})\|, \|every(\{m\})(\{j, m\}) \wedge \neg every(\{j\})(\{j, m\})\|\})$$

$$= m_{0.5}(\{0\})$$

$$= 0 \quad \text{by (13)}$$

Finally, by (14),

$$\|(16)\| = \|(18)\| = \int_0^1 \|every_\gamma(\{m\})(TALL) \wedge \neg every_\gamma(\{j\})(TALL)\| d\gamma$$

$$= 0 \times (0.4 - 0) + 0 \times (1 - 0.4)$$

$$= 0$$

which is as desired. Note that if we had included $\{m\}$ as an admissible complete specification for $0.4 < \gamma \leq 1$, then we would have got $\|(16)\| = 0.3$, contrary to our intuition. The above computation shows that MGM is able to correct the flaw of FT.

3.2 Some Properties of MGM

The modification of the definition of $T_\gamma(X)$ as shown in (17) may incur a cost in that some nice properties of Glöckner's original theory may be lost. By scrutinizing the proofs of the various lemmas and theorems in [4], one can find that the important properties of the orginal theory introduced in Subsection 2.2 are not affected by the modification with two exceptions, namely under MGM the QFM represented by M does not commute with internal meet and functional application. This means, for example, that when we evaluate[13]

$$\| Q_\gamma(X_1 \cap X_2) \| \tag{19}$$

for a particular γ, the result of first precisifying X_1 and X_2 and then intersecting the resultant crisp sets, i.e.

$$m_{0.5}(\{ \| Q(Y_1 \cap Y_2) \| : Y_1 \in T_\gamma(X_1), Y_2 \in T_\gamma(X_2)\}) \tag{20}$$

may be different from the result of first intersecting X_1 and X_2 and then precisifying the resultant fuzzy set, i.e.

$$m_{0.5}(\{ \| Q(Y) \| : Y \in T_\gamma(X_1 \cap X_2)\}) \tag{21}$$

because $\{Y_1 \cap Y_2: Y_1 \in T_\gamma(X_1), Y_2 \in T_\gamma(X_2)\}$ may not be equal to $\{Y: Y \in T_\gamma(X_1 \cap X_2)\}$. The crux of the problem is that the intersection of two sets each representing an admissible complete specification may not be a set representing an admissible complete specification. For instance, while {a, b} represents an admissible complete specification for the set X_1 = {1/a, 0.9/b, 0.8/c} and {b, c} represents an admissible complete specification for the set X_2 = {0.5/a, 0.6/b, 0.7/c}, {a, b} ∩ {b, c} = {b} represents an inadmissible complete specification for $X_1 \cap X_2$ = {0.5/a, 0.6/b, 0.7/c}. The same problem can be said of functional application for an arbitrary function.

Is this a defect of MGM? Not necessarily. The essence of MGM is to deny the truth functionality of BOs and other arbitrary functions when applied to vague sets. Under MGM, when evaluating the truth value of a vague statement involving BOs or other arbitrary functions, we never apply the BOs or functions to the vague arguments directly because such application is undefined. Instead, we always proceed by first precisifying the vague arguments and then applying the BOs or functions to the resultant crisp arguments. This means, for example, that when evaluating (19) we always do (20), never (21), and so the problem that (20) ≠ (21) simply does not arise. Thus, we may say that MGM has preserved the essential nice properties of Glöckner's original theory.

Note that MGM also has another nice property. Suppose the membership degrees with respect to the vague sets $X_1, ... X_n$ in a model are restricted to {0, 1, 0.5} and the truth values of a semi-fuzzy quantifier Q applied to any n crisp arguments are also restricted to {0, 1, 0.5}. Then for $0 < \gamma \leq 1$, we must have $\{ \| Q(Y_1, \dots Y_n) \| : Y_1 \in T_\gamma(X_1), \dots Y_n \in T_\gamma(X_n)\}$ equal to any one of the following: {0}, {1}, {0.5}, {0, 1}, {0, 0.5}, {1, 0.5}, {0, 1, 0.5}. By (12) and (13), we have $\| Q_\gamma(X_1, \dots X_n) \|$ equal to 0, 1 or 0.5 according as $\{ \| Q(Y_1, \dots Y_n) \| : Y_1 \in T_\gamma(X_1), \dots Y_n \in T_\gamma(X_n)\}$ contains only 0, only 1 or otherwise. Then by (14), we have $\| M(Q)(X_1, \dots X_n) \|$ also restricted to {0, 1, 0.5}. So in this case MGM gives us the same result as that obtained by the supervaluation method if we use 0.5 to represent the truth value gap. MGM is thus

[13] To simplify notation, in what follows I use the same symbol "∩" to denote the intersection operation of crisp sets and vague sets. Under FT, the vague version of "∩" may be defined based on the BO "∧".

indeed a generalization of the supervaluation method and provides us with the flexibility in determining how we should model vagueness.

4 Iterated VQs

According to [9], a sentence containing both subject and object(s) can be viewed as containing a polyadic quantifier. There is an important type of polyadic quantifiers, called iterated quantifiers, that can be represented by a tripartite structure with one of its arguments containing another tripartite structure. For example, the sentence

Every boy loves every girl. (22)

may be seen as containing the iterated quantifier "(*every* ... *every*)" and can be represented by the following tripartite structure:

$$every(\text{BOY})(\{x: every(\text{GIRL})(\{y: \text{LOVE}(x, y)\})\}) \quad (23)$$

which, in daily language, means "Every boy x is such that for every girl y, x loves y". Based on the above expression, one can then evaluate the truth value of (22) with respect to any model according to the truth condition of "*every*".

MGM is readily applicable to iterated VQs[14]. Consider the following sentence:

Almost every boy met about 10 girls. (24)

with respect to the following model:

M2 BOY = {a, b, c, d, e}

x	a	b	c	d	e
|GIRL ∩ {y: MEET(x, y)}|	10	9	11	13	8

For computational purpose, suppose we use (7) as the MDF for "(*about 10*)" and the following MDF for "(*almost every*)" (where ε represents an infinitesimal positive magnitude):

$$\| (almost\ every)(\text{A})(\text{B}) \| = \text{T}_{-0.4,\, -0.2,\, -\varepsilon,\, -\varepsilon}(|\text{A} \cap \text{B}| \,/\, |\text{A}| - 1) \quad (25)$$

To evaluate ‖(24)‖, we first write (24) as the following tripartite structure:

$$(almost\ every)(\text{BOY})(\{x: (about\ 10)(\text{GIRL})(\{y: \text{MEET}(x, y)\})\}) \quad (26)$$

In the above, {x: (*about 10*)(GIRL)({y: MEET(x, y)})} denotes the set of those who met about 10 girls. For convenience, let's call this set X. Since X is a vague set, we cannot evaluate ‖(26)‖ directly. To facilitate further computation, we need to determine this vague set first. According to (7), for each x, ‖(*about 10*)(GIRL)({y: MEET(x, y)})‖ depends on the input |GIRL ∩ {y: MEET(x, y)}| – 10. By

14 For simplicity, here I only consider iterated VQs composed of 2 VQs. It is not difficult to generalize the theory to iterated VQs composed of more than 2 VQs.

substituting the data given in M2 into (7) for each x, we can determine the following vague set: X = {1/a, 1/b, 1/c, 0.33/d, 0.67/e}.

We then use MGM to evaluate $\|(26)\|$. For $0 < \gamma \leq 0.33$, we have X_γ^{min} = {a, b, c, e}, X_γ^{max} = {a, b, c, e} and $T_\gamma(X)$ = {{a, b, c, e}} because {a, b, c, e} represents an admissible complete specification. Then, we have $\|$(*almost every*)$_\gamma$(BOY)(X)$\|$ = $m_{0.5}$({$\|$(*almost every*)$_\gamma$(BOY)({a, b, c, e})$\|$}) = $m_{0.5}$({1}) = 1.

For $0.33 < \gamma \leq 1$, we have X_γ^{min} = {a, b, c}, X_γ^{max} = {a, b, c, d, e} and $T_\gamma(X)$ = {{a, b, c}, {a, b, c, e}, {a, b, c, d, e}} because {a, b, c}, {a, b, c, e} and {a, b, c, d, e} all represent admissible complete specifications (Note that {a, b, c, d} represents an inadmissible complete specification and is thus excluded). Then, we have $\|$(*almost every*)$_\gamma$(BOY)(X)$\|$ = $m_{0.5}$({$\|$(*almost every*)$_\gamma$(BOY)({a, b, c})$\|$, $\|$(*almost every*)$_\gamma$(BOY)({a, b, c, e})$\|$, $\|$(*almost every*)$_\gamma$(BOY)({a, b, c, d, e})$\|$}) = $m_{0.5}$({0, 1}) = 0.5.

Finally, by (14),

$$\|(24)\| = \|(26)\| = \int_0^1 \|(\textit{almost every})_\gamma(\mathrm{BOY})(X)\| \, d\gamma$$
$$= 1 \times (0.33 - 0) + 0.5 \times (1 - 0.33)$$
$$= 0.67$$

Note that the above calculation has been greatly simplified because in (24), "boy", "girls" and "met" are all represented by crisp predicates. In general, given a sentence with iterated VQs, we first express it in the following form:

$$Q_1(A_1)(\{x: Q_2(A_2)(\{y: B(x, y)\})\}) \quad (27)$$

Then for each possible x, we determine {y: B(x, y)}, which may be a vague set, and $\|Q_2(A_2)(\{y: B(x, y)\})\|$ by MGM. By doing so, we will obtain the following set: $\{x: Q_2(A_2)(\{y: B(x, y)\})\} = \{\|Q_2(A_2)(\{y: B(x_i, y)\})\|/x_i, \ldots\}$, where x_i ranges over all possible xs. Finally, we can evaluate $\|Q_1(A_1)(\{x: Q_2(A_2)(\{y: B(x, y)\})\})\|$ by MGM.

5 Conclusion

In this paper, I have discussed the merits and demerits of the FT and ST approaches to vagueness and have proposed MGM as a model for VQs. This model inherits certain desirable properties of Glöckner's framework in [4]. It is also able to distinguish different degrees of vagueness and is thus useful for practical applications. Moreover, this model has overcome a demerit commonly found in FT frameworks, i.e. it yields correct results for penumbral connections. I have also shown that Glöckner's original method in fact includes a process reminiscent of the precisification process of ST. This provides a plausible way to combine FT and ST, the two main competing theories for vagueness.

Nevertheless, this paper has just concentrated on one particular QFM represented by M. As a matter of fact, Glöckner and other scholars have proposed other possible QFMs in [1] and [4-5] which I have not had the chance to discuss in this paper. It would be instructive to consider how these QFMs can be modified to suit the requirement of ST and what properties of Glöckner's original theory are preserved under the modification and would thus be a possible direction for future studies.

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