RELATIONAL SYLLOGISMS WITH COMPARATIVE RELATIONS

Ka-Fat Chow

Abstract

This paper presents a method for deriving valid relational syllogisms with comparative relations, called DMcr. We first introduce some basic notions and notation for comparative relations, predicates, quantifiers, scope dominance, scopelessness and syllogisms. We next describe DMcr in detail and prove that DMcr is sound. We then discuss how we can extend DMcr so as to derive more valid syllogisms, including the use of existential assumptions, and restriction to finite domains. Finally, we point to several possible directions for future studies on relational syllogisms.

Keywords: Relational syllogisms, simple syllogisms, comparative relations, scope

1. Introduction

Relational syllogisms refer to syllogisms containing n-ary predicates (where $n > 1$) in the premises and/or conclusion which make the sentence structure more complicated (with object, oblique argument, relative clause), as opposed to simple syllogisms, whose premises and conclusion contain only unary predicates. Given this complexity, relational syllogisms have been less studied than simple syllogisms, and among those studied, the studies have been focused on relational syllogisms with the classical quantifiers. There are few studies of relational syllogisms with non-classical quantifiers.

In Chow 2022, two new Derivation Methods for deriving relational syllogisms have been introduced. Under the new Methods, relational syllogisms are not derived directly from axioms and/or inference rules as in Keene 1969, Pratt-Hartmann and Moss 2009, Moss 2010, Moss 2011, Ivanov and Vakarelov 2012, Pratt-Hartmann 2013, Kruckman and Moss 2021, to name just a few. Instead, known valid simple syllogisms (with classical or non-classical quantifiers) are used as a starting point to derive valid relational

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syllogisms by applying certain validity-preserving operations on these simple syllogisms. In this way, the Methods are guaranteed to be sound.

While the Derivation Methods are applicable to relational syllogisms with general binary predicates, it has also been pointed out in Chow 2022 that quantified statements with comparative relations such as "be taller than" satisfy certain syllogistic patterns that quantified statements with general binary predicates do not satisfy. The following is an example of a valid syllogism with comparative relations:

Example 1. Every runner is shorter than most basketball players. Every swimmer is shorter than a runner. Therefore, every swimmer is shorter than most basketball players.

Note that the above syllogism remains valid if "is shorter than" is replaced by another comparative relation such as "is in front of", but becomes invalid if it is replaced by a general binary predicate such as "likes". Thus, the study on comparative relations will enable us to identify more valid relational syllogistic patterns.

The main aim of this paper is to extend the Derivation Methods and results in Chow 2022 to relational syllogisms with comparative relations. This paper is organized as follows. We first introduce some basic notions and notation for comparative relations, predicates, quantifiers, scope dominance, scopelessness and syllogisms. We next describe a derivation method for relational syllogisms with comparative relations, called DMcr, in detail, and prove the soundness of the method. We then discuss how we can further identify valid relational syllogisms with comparative relations by using existential assumptions and restricting the domains to finite ones. Finally, we point to several possible directions for future studies on relational syllogisms.

2. Basic Notions and Notation

2.1. *Comparative Relations*

We first introduce some notation. In this paper, we use the small-case letter r to represent a generic binary predicate. The denotation of r, written $[\![r]\!]$ in this paper, is a set of ordered pairs. Since a set can also be seen as a characteristic function, we will use either $(\mathtt{x}, \mathtt{y}) \in [\![r]\!]$ or $[\![r]\!](\mathtt{x}, \mathtt{y}) = 1$ to express the fact that $\mathtt{x}$ and $\mathtt{y}$ stand in the relation $[\![r]\!]$, where $\mathtt{x}$ and $\mathtt{y}$ are two members of the domain of discourse $\mathtt{U}$[1].

The negation of r will be represented by $\neg r$, with its denotation given as follows:

$$\text{For all } \mathtt{x}, \mathtt{y} \in \mathtt{U}, [\![\neg r]\!](\mathtt{x}, \mathtt{y}) = 1 \text{ iff } [\![r]\!](\mathtt{x}, \mathtt{y}) = 0 \tag{1}$$

The converse of r will be represented by r^{-1}, with its denotation given as follows:

$$\text{For all } \mathtt{x}, \mathtt{y} \in \mathtt{U}, [\![r^{-1}]\!](\mathtt{x}, \mathtt{y}) = 1 \text{ iff } [\![r]\!](\mathtt{y}, \mathtt{x}) = 1 \tag{2}$$

From (1) and (2), one can easily deduce the following "double negation law" and "double converse law":

$$\text{For all binary predicates } r, [\![\neg(\neg r)]\!] = [\![r]\!]; [\![(r^{-1})^{-1}]\!] = [\![r]\!] \tag{3}$$

The combined negation and converse of r will be represented by $\neg r^{-1}$. Note that this notation can be understood to be either $\neg(r^{-1})$ or $(\neg r)^{-1}$, which mean the same thing, because for any $\mathtt{x}, \mathtt{y} \in \mathtt{U}$, $[\![\neg(r^{-1})]\!](\mathtt{x}, \mathtt{y}) = 1$ iff $[\![r^{-1}]\!](\mathtt{x}, \mathtt{y}) = 0$ iff $[\![r]\!](\mathtt{y}, \mathtt{x}) = 0$ iff $[\![\neg r]\!](\mathtt{y}, \mathtt{x}) = 1$ iff $[\![(\neg r)^{-1}]\!](\mathtt{x}, \mathtt{y}) = 1$.

Having fixed the notation for binary predicates, we can now define strict comparative relations. Borrowing ideas from Keene 1969, we define strict comparative relations to be binary predicates with the properties of asymmetry, transitivity and counter-transitivity, which are defined as follows. Let r be a binary predicate, then

(i) r is asymmetric iff

$$\text{for all } \mathtt{x}, \mathtt{y} \in \mathtt{U}, [\![r]\!](\mathtt{x}, \mathtt{y}) = 1 \Rightarrow [\![\neg r^{-1}]\!](\mathtt{x}, \mathtt{y}) = 1 \tag{4}$$

(ii) r is transitive iff

$$\text{for all } \mathtt{x}, \mathtt{y}, \mathtt{z} \in \mathtt{U}, ([\![r]\!](\mathtt{x}, \mathtt{y}) = 1 \wedge [\![r]\!](\mathtt{y}, \mathtt{z}) = 1) \Rightarrow [\![r]\!](\mathtt{x}, \mathtt{z}) = 1 \tag{5}$$

(iii) r is counter-transitive iff

$$\text{for all } \mathtt{x}, \mathtt{y}, \mathtt{z} \in \mathtt{U}, ([\![\neg r]\!](\mathtt{x}, \mathtt{y}) = 1 \wedge [\![\neg r]\!](\mathtt{y}, \mathtt{z}) = 1) \Rightarrow [\![\neg r]\!](\mathtt{x}, \mathtt{z}) = 1 \tag{6}$$

Corresponding to each strict comparative relation, there is a non-strict comparative relation. Where necessary, the corresponding strict and non-strict

1. In this paper, an object in a model is represented in the `courier font`, unless it is the denotation of an expression and is thus placed in $[\![\ ]\!]$, or a special object with a dedicated symbol such as the empty set with the symbol $\emptyset$.

comparative relations will be represented more specifically by $r^>$ and $r^\geq$, respectively. The denotation of $r^\geq$ is given as follows:

$$\text{For all } \mathtt{x}, \mathtt{y} \in \mathtt{U}, [\![r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1 \text{ iff } [\![\neg r^{>-1}]\!](\mathtt{x}, \mathtt{y}) = 1 \tag{7}$$

From the definitions given above, we can deduce a number of properties of non-strict comparative relations and relations between the two types of comparative relations, which are summarized in the following proposition.

Proposition 1. Let $r^>$ be a strict comparative relation, and $r^\geq$ be the corresponding non-strict comparative relation. Then for all $\mathtt{x}, \mathtt{y}, \mathtt{z} \in \mathtt{U}$,
(i) ($r^>$ entails $r^\geq$) $[\![r^>]\!](\mathtt{x}, \mathtt{y}) = 1 \Rightarrow [\![r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1$
(ii) (Linearity of $r^\geq$) $[\![\neg r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1 \Rightarrow [\![(r^{\geq-1}]\!](\mathtt{x}, \mathtt{y}) = 1$
(iii) (Transitivity of $r^\geq$) $([\![r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1 \wedge [\![r^\geq]\!](\mathtt{y}, \mathtt{z}) = 1) \Rightarrow [\![r^\geq]\!](\mathtt{x}, \mathtt{z}) = 1$
(iv) (Counter-transitivity of $r^\geq$) $([\![\neg r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1 \wedge [\![\neg r^\geq]\!](\mathtt{y}, \mathtt{z}) = 1) \Rightarrow [\![\neg r^\geq]\!](\mathtt{x}, \mathtt{z}) = 1$
(v) (Mixed transitivity 1) $([\![r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1 \wedge [\![r^>]\!](\mathtt{y}, \mathtt{z}) = 1) \Rightarrow [\![r^>]\!](\mathtt{x}, \mathtt{z}) = 1$
(vi) (Mixed transitivity 2) $([\![r^>]\!](\mathtt{x}, \mathtt{y}) = 1 \wedge [\![r^\geq]\!](\mathtt{y}, \mathtt{z}) = 1) \Rightarrow [\![r^>]\!](\mathtt{x}, \mathtt{z}) = 1$

Proof. (i) Assume $[\![r^>]\!](\mathtt{x}, \mathtt{y}) = 1$. Then by asymmetry of $r^>$, this entails $[\![\neg r^{>-1}]\!](\mathtt{x}, \mathtt{y}) = 1$, which by (7) is equivalent to $[\![r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1$.
(ii) Assume $[\![\neg r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1$, which by (7), (3) and (2) is equivalent to $[\![r^>]\!](\mathtt{y}, \mathtt{x}) = 1$. Then by (i) proved above, it follows that $[\![r^\geq]\!](\mathtt{y}, \mathtt{x}) = 1$, which is equivalent to $[\![r^{\geq-1}]\!](\mathtt{x}, \mathtt{y}) = 1$.
(iii) Assume $[\![r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1$ and $[\![r^\geq]\!](\mathtt{y}, \mathtt{z}) = 1$, which by (7) and (2) are equivalent to $[\![\neg r^>]\!](\mathtt{y}, \mathtt{x}) = 1$ and $[\![\neg r^>]\!](\mathtt{z}, \mathtt{y}) = 1$, respectively. Then, by counter-transitivity of $r^>$, it follows that $[\![\neg r^>]\!](\mathtt{z}, \mathtt{x}) = 1$, which by (2) and (7) is equivalent to $[\![r^\geq]\!](\mathtt{x}, \mathtt{z}) = 1$.
(iv) The proof is analogous to that of (iii).
(v) Assume $[\![r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1$ and $[\![r^>]\!](\mathtt{y}, \mathtt{z}) = 1$. Further assume that $[\![r^>]\!](\mathtt{x}, \mathtt{z}) = 0$. By (7) and (2) the first assumption is equivalent to $[\![\neg r^>]\!](\mathtt{y}, \mathtt{x}) = 1$. By (1) the third assumption is equivalent to $[\![\neg r^>]\!](\mathtt{x}, \mathtt{z}) = 1$. Then by counter-transitivity of $r^>$, the first and third assumptions imply that $[\![\neg r^>]\!](\mathtt{y}, \mathtt{z}) = 1$, which contradicts the second assumption. This contradiction shows that the third assumption is false, and so we have $[\![r^>]\!](\mathtt{x}, \mathtt{z}) = 1$.
(vi) The proof is analogous to that of (v). □

One can easily see that if r possesses the property of asymmetry, linearity, transitivity or counter-transitivity, then its converse r^{-1} also possesses that property. Thus, if $r^>$ and $r^\geq$ are replaced by $r^{>-1}$ and $r^{\geq-1}$ respectively in Proposition 1, the proposition remains valid. Moreover, by (7), the negation of a strict comparative relation is a non-strict comparative relation that is linear, transitive and counter-transitive, and the negation of a non-strict comparative relation is a strict comparative relation that is asymmetric, transitive and counter-transitive.

Furthermore, corresponding to each strict comparative relation r, we may also define an equative comparative relation $r^{=}$ with its denotation given as below:

$$\text{For all } \mathtt{x}, \mathtt{y} \in \mathtt{U}, [\![r^{=}]\!](\mathtt{x}, \mathtt{y}) = 1 \text{ iff } [\![r^{\geq}]\!](\mathtt{x}, \mathtt{y}) = 1 \wedge [\![\neg r^{>}]\!](\mathtt{x}, \mathtt{y}) = 1 \quad (8)$$

Then since $[\![r^{>}]\!](\mathtt{x}, \mathtt{y}) = 1 \vee ([\![r^{\geq}]\!](\mathtt{x}, \mathtt{y}) = 1 \wedge [\![\neg r^{>}]\!](\mathtt{x}, \mathtt{y}) = 1)$ is equivalent to $([\![r^{>}]\!](\mathtt{x}, \mathtt{y}) = 1 \vee [\![r^{\geq}]\!](\mathtt{x}, \mathtt{y}) = 1) \wedge ([\![r^{>}]\!](\mathtt{x}, \mathtt{y}) = 1 \vee [\![\neg r^{>}]\!](\mathtt{x}, \mathtt{y}) = 1)$, which by Proposition 1(i) is equivalent to $[\![r^{\geq}]\!](\mathtt{x}, \mathtt{y}) = 1$, we have thus shown that the following relation holds:

$$\text{For all } \mathtt{x}, \mathtt{y} \in \mathtt{U}, [\![r^{\geq}]\!](\mathtt{x}, \mathtt{y}) = 1 \text{ iff } [\![r^{>}]\!](\mathtt{x}, \mathtt{y}) = 1 \vee [\![r^{=}]\!](\mathtt{x}, \mathtt{y}) = 1 \quad (9)$$

The above relation can be used as an alternative specification of $[\![r^{\geq}]\!]$. However, since the focus of this paper is on strict and non-strict comparative relations, we will not further explore the properties of equative comparative relations.

Most examples of strict comparative relations can be found among comparative adjectives. A typical example is "taller than". According to (7) or (9), the corresponding non-strict comparative relation is "not shorter than" or "taller than or as tall as". Apart from comparative adjectives, some prepositions are also examples of strict comparative relations. An example is "in front of" when used to describe objects that can be seen as being arranged on a line. The corresponding non-strict comparative relation is "not behind" or "in front of or in the same position as".

2.2. *Predicates and Quantifiers*

In this paper, we will adopt a notation for quantified statements that is similar to Chow 2022 and based on the Generalized Quantifier Theory (GQT) as presented in, say Barwise and Cooper 1981, Peters and Westerståhl 2006 and Keenan and Westerståhl 2011. Under GQT, a quantifier is seen as a second-order predicate with ordinary (first-order) predicate(s) as its argument(s), and a quantified statement is made up of a quantifier plus its argument(s). Quantifiers can be classified according to the number and type of argument(s) required. A type $\langle 1 \rangle$ quantifier, represented generically as Q in this paper, requires one unary predicate. The denotation of a type $\langle 1 \rangle$

quantifier is thus a set of sets, or equivalently a characteristic function with a set as its argument. A type $\langle 1, 1\rangle$ quantifier, commonly called "determiner" in the GQT literature and so represented generically as D in this paper, requires two unary predicates as its arguments, which will be called the left (first) and the right (second) arguments. The denotation of a determiner is thus a set of ordered pairs of sets, or equivalently a characteristic function with two sets as its arguments.

In this paper, we will use small-case letters in the beginning of the alphabet list, namely a, b, c, ... to represent unary predicates. Using the notation of Chow 2022, we will use nA, nE, mI and mO to represent the four types of numerical determiners studied by Murphree in Murphree 1991, Murphree 1993, Murphree 1997, Murphree 1998, namely "at least all but n", "at most n", "at least m" and "at most all but m", respectively[2], where n is an appropriate non-negative integer and m is an appropriate positive integer. The denotation of these determiners is given below:

$$[\![nA]\!](\mathsf{a},\mathsf{b}) = 1 \text{ iff } |\mathsf{a} - \mathsf{b}| \leq n. \tag{10}$$

$$[\![nE]\!](\mathsf{a},\mathsf{b}) = 1 \text{ iff } |\mathsf{a} \cap \mathsf{b}| \leq n. \tag{11}$$

$$[\![mI]\!](\mathsf{a},\mathsf{b}) = 1 \text{ iff } |\mathsf{a} \cap \mathsf{b}| \geq m. \tag{12}$$

$$[\![mO]\!](\mathsf{a},\mathsf{b}) = 1 \text{ iff } |\mathsf{a} - \mathsf{b}| \geq m. \tag{13}$$

From the above denotation, one can easily see that these four types of numerical determiners are in fact extensions of the four classical quantifiers in that "every" = "at least all but 0", "no" = "at most 0", "some" = "at least 1" and "not every" = "at most all but 1". While the classical quantifiers can be represented as $0A$, $0E$, $1I$ and $1O$, this paper will follow the traditional way by representing them simply as A, E, I and O, as no confusion will arise if they are represented in this simpler way.

Some examples in this paper will involve proportional determiners. Following Johnson 1994, we will use $\geq_f$, $>_f$, $\leq_f$ and $<_f$, where f is a fraction such that $0 < f < 1$, to represent the proportional determiners "at least f of", "more than f of", "at most f of" and "less than f of", respectively. The denotation of $\geq_f$ is given below:

$$[\![\geq_f]\!](\mathsf{a},\mathsf{b}) = 1 \text{ iff } |\mathsf{a} \cap \mathsf{b}| \geq f|\mathsf{a}|. \tag{14}$$

2. "At least all but n" is usually (and more naturally) expressed as "all but at most n", whereas "at most all but m" is usually (and more naturally) expressed as "at least m ... not".

The denotation of other proportional determiners is similar. Following the convention of GQT, we will assume that "most" is the same as "more than a half of" and use a special symbol M to represent "most". This means that we have $M = >_{\frac{1}{2}}$.

Note that a determiner plus a unary predicate in its left argument can also be seen as a type $\langle 1 \rangle$ quantifier because this structure requires only one unary predicate to make up a quantified statement. Hence, a quantified statement with a determiner D plus two unary predicates a and b, which is usually written as $D(a, b)$, can also be seen as made up of a type $\langle 1 \rangle$ quantifier $D(a)$ plus one unary predicate b and thus written as $D(a)(b)$.

When a quantified statement contains a binary predicate, including the comparative relations discussed in this paper, the situation is a bit more complicated. In the GQT literature, type $\langle 1 \rangle$ quantifiers can be seen as "arity reducers" which, when combined with an n-ary predicate, will reduce that predicate to an $n-1$-ary predicate[3]. Hence, when the type $\langle 1 \rangle$ quantifier $I(a)$ combines with the binary predicate r, the result $I(a)(r)$ is a unary predicate. When the type $\langle 1 \rangle$ quantifier $A(b)$ combines with this unary predicate, the result $A(b)[I(a)(r)]$, which may also be written as $A[b, I(a)(r)]$, is a quantified statement.

We stipulate that $Q(r)$ should denote a meaning such that Q acts as the "subject" of r semantically. Thus, the denotation of $Q(r)$ is given below:

$$[\![Q(r)]\!] = \{\mathtt{y} : [\![Q]\!](\{\mathtt{x} : [\![r]\!](\mathtt{x}, \mathtt{y}) = 1\}) = 1\} \tag{15}$$

For example, if r represents the comparative relation "be taller than", then $A(a)(r)$ means "that which every a is taller than it". A statement with an "object" (as well as a "subject") of r like "Some b is such that every a is taller than it" will be represented under this notation by $I[b, A(a)(r)]$, while a statement with a headless relative clause like "Every individual that every a is taller than it is a b" is represented by $A[A(a)(r), b]$.

In case we wish to express a unary predicate in which Q acts as the "object" of r semantically, we may use the converse of r, namely r^{-1}. By changing r in (15) to r^{-1} and then using (2), the denotation of $Q(r^{-1})$ can be written as

$$[\![Q(r^{-1})]\!] = \{\mathtt{x} : [\![Q]\!](\{\mathtt{y} : [\![r]\!](\mathtt{x}, \mathtt{y}) = 1\}) = 1\} \tag{16}$$

For example, if r represents the comparative relation "be taller than", then r^{-1} represents "be shorter than". Hence, $A(a)(r^{-1})$ means "that which every

3. A quantified statement can be seen as a 0-ary predicate.

a is shorter than it", or equivalently, "that which is taller than every a", where "every a" acts as the "object" of "be taller than", and $I[b, A(a)(r^{-1})]$ means "Some b is taller than every a".

2.3. *Some Properties and Operations of Determiners*

In this subsection, we introduce some special properties and operations of determiners which will be useful in later sections. The first property is conservativity, which is defined as follows: a determiner D is conservative iff for all $\mathsf{a}, \mathsf{b} \subseteq \mathsf{U}$,

$$[\![D]\!](\mathsf{a}, \mathsf{b}) = 1 \Leftrightarrow [\![D]\!](\mathsf{a}, \mathsf{a} \cap \mathsf{b}) = 1 \tag{17}$$

It can be shown that all the determiners introduced above are conservative.

The second property is right increasing monotonicity which is defined as follows: a determiner D is right increasing iff for all $\mathsf{a}, \mathsf{b}_1, \mathsf{b}_2 \subseteq \mathsf{U}$ such that $\mathsf{b}_1 \subseteq \mathsf{b}_2$,

$$[\![D]\!](\mathsf{a}, \mathsf{b}_1) = 1 \Rightarrow [\![D]\!](\mathsf{a}, \mathsf{b}_2) = 1 \tag{18}$$

It can be shown that the determiners nA, mI, $\geq_f$ and $>_f$ (including M) introduced above are right increasing.

There is a useful proposition that relates right increasing monotonicity and Proposition 1(i).

Proposition 2. Let D_1 and D_2 be right increasing determiners, a, b be unary predicates, $r^>$ and $r^\geq$ be corresponding strict and non-strict comparative relations. Then $[\![D_1[a, D_2(b)(r^>)]]\!] = 1 \Rightarrow [\![D_1[a, D_2(b)(r^\geq)]]\!] = 1$.

Proof. By Proposition 1(i), for all $\mathtt{x}, \mathtt{y} \in \mathtt{U}$, we have $[\![r^>]\!](\mathtt{x}, \mathtt{y}) = 1 \Rightarrow [\![r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1$. Therefore, for any $\mathtt{y}$, we have $\{\mathtt{x} : [\![r^>]\!](\mathtt{x}, \mathtt{y}) = 1\} \subseteq \{\mathtt{x} : [\![r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1\}$. Since D_2 is right increasing, we have $[\![D_2(b)]\!](\{\mathtt{x} : [\![r^>]\!](\mathtt{x}, \mathtt{y}) = 1\}) = 1 \Rightarrow [\![D_2(b)]\!](\{\mathtt{x} : [\![r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1\}) = 1$. But since $\mathtt{y}$ is arbitrary, we have $\{\mathtt{y} : [\![D_2(b)]\!](\{\mathtt{x} : [\![r^>]\!](\mathtt{x}, \mathtt{y}) = 1\}) = 1\} \subseteq \{\mathtt{y} : [\![D_2(b)]\!](\{\mathtt{x} : [\![r^\geq]\!](\mathtt{x}, \mathtt{y}) = 1\}) = 1\}$, which by (15) can be rewritten as $[\![D_2(b)(r^>)]\!] \subseteq [\![D_2(b)(r^\geq)]\!]$. Since D_1 is also right increasing, we have $[\![D_1[a, D_2(b)(r^>)]]\!] = 1 \Rightarrow [\![D_1[a, D_2(b)(r^\geq)]]\!] = 1$, which is the desired result. □

The third property is right positivity, which is defined as follows: a determiner D is right positive iff for all $\mathsf{a} \subseteq \mathsf{U}$,

$$[\![D]\!](\mathsf{a}, \emptyset) = 0 \tag{19}$$

It can be shown that the determiners mI and $>_f$ (including M) introduced above are right positive.

The last two properties are symmetry and contrapositivity, which are defined as follows: a determiner D is symmetric iff for all $\mathtt{a}, \mathtt{b} \subseteq \mathtt{U}$,

$$[\![D]\!](\mathtt{a}, \mathtt{b}) = 1 \Leftrightarrow [\![D]\!](\mathtt{b}, \mathtt{a}) = 1 \tag{20}$$

A determiner D is contrapositive iff for all $\mathtt{a}, \mathtt{b} \subseteq \mathtt{U}$ (in what follows, $\neg\mathtt{a} = \mathtt{U} - \mathtt{a}$),

$$[\![D]\!](\mathtt{a}, \mathtt{b}) = 1 \Leftrightarrow [\![D]\!](\neg\mathtt{b}, \neg\mathtt{a}) = 1 \tag{21}$$

It can be shown that the determiners nE and mI are symmetric while the determiners nA and mO are contrapositive.

We next introduce a useful operation on determiners, namely witness sets. Let D be a determiner and a be a unary predicate. Then ω is a witness set of $D(a)$ iff

$$\omega \subseteq [\![a]\!] \wedge [\![D]\!]([\![a]\!], \omega) = 1 \tag{22}$$

Intuitively, a witness set of $D(a)$ is a set that can reflect the meaning of $D(a)$. For example, let $[\![a]\!] = \{\mathtt{u}, \mathtt{v}, \mathtt{w}\}$. Then the set of witness sets of $I(a)$ can be computed as follows:

$$\begin{aligned} & \{\omega : \omega \subseteq [\![a]\!] \wedge [\![I]\!]([\![a]\!], \omega) = 1\} \\ = \; & \{\omega : \omega \subseteq [\![a]\!] \wedge |[\![a]\!] \cap \omega| \geq 1\} \\ = \; & \{\{\mathtt{u}\}, \{\mathtt{v}\}, \{\mathtt{w}\}, \{\mathtt{u}, \mathtt{v}\}, \{\mathtt{u}, \mathtt{w}\}, \{\mathtt{v}, \mathtt{w}\}, \{\mathtt{u}, \mathtt{v}, \mathtt{w}\}\} \end{aligned}$$

One can see that every set in the above set contains at least one element of $[\![a]\!]$ and so reflects the meaning of $I(a)$.

There are two important propositions concerning witness sets and the properties of determiners introduced above.

PROPOSITION 3. Let D be a conservative right increasing determiner, and a, b be unary predicates. Then $[\![D]\!]([\![a]\!], [\![b]\!]) = 1$ iff there exists ω which is a witness set of $D(a)$ such that $\omega \subseteq [\![b]\!]$.[4]

Proof. First assume that $[\![D]\!]([\![a]\!], [\![b]\!]) = 1$. Since D is conservative, by (17), we have $[\![D]\!]([\![a]\!], [\![a]\!] \cap [\![b]\!]) = 1$. Then, $[\![a]\!] \cap [\![b]\!]$ is the required witness set

4. This proposition is the same as Proposition C11(i) in Barwise and Cooper 1981. But since we have formulated the definition of witness sets a bit differently (we do not use the notion "living on" as in Barwise and Cooper 1981), we provide the proof of this proposition that suits the definitions in this paper.

because $[\![a]\!] \cap [\![b]\!] \subseteq [\![a]\!]$ and $[\![D]\!]([\![a]\!], [\![a]\!] \cap [\![b]\!]) = 1$, satisfying (22) and $[\![a]\!] \cap [\![b]\!] \subseteq [\![b]\!]$. Next assume that there exists ω which is a witness set of $D(a)$ such that $\omega \subseteq [\![b]\!]$. By (22), we have $[\![D]\!]([\![a]\!], \omega) = 1$. Moreover, since D is right increasing, we have $[\![D]\!]([\![a]\!], [\![b]\!]) = 1$. □

PROPOSITION 4. Let D be a right positive determiner and a be a unary predicate. Then every witness set of $D(a)$ is non-empty.

Proof. Suppose $\emptyset$ is a witness set of $D(a)$. Then by (22), we have $[\![D]\!]([\![a]\!], \emptyset) = 1$. But since D is right positive, by (19), we also have $[\![D]\!]([\![a]\!], \emptyset) = 0$. This contradiction shows that $\emptyset$ cannot be a witness set of $D(a)$. □

Negation constitutes another important type of operations on determiners. According to Peters and Westerståhl 2006 and Keenan and Westerståhl 2011, a determiner D has two types of negation: outer negation (also called complement), represented by $\neg D$, and inner negation (also called postcomplement), represented by $D^{\sim}$, as well as a dual, which is the combination of the outer and inner negations and is represented by D^d. The denotations of these three notions are given below. Let D be a determiner, then for all $\mathsf{a}, \mathsf{b} \subseteq \mathsf{U}$,

$$[\![\neg D]\!](\mathsf{a}, \mathsf{b}) = 1 \text{ iff } [\![D]\!](\mathsf{a}, \mathsf{b}) = 0 \tag{23}$$

$$[\![D^{\sim}]\!](\mathsf{a}, \mathsf{b}) = 1 \text{ iff } [\![D]\!](\mathsf{a}, \neg\mathsf{b}) = 1 \tag{24}$$

$$[\![D^{d}]\!](\mathsf{a}, \mathsf{b}) = 1 \text{ iff } [\![D]\!](\mathsf{a}, \neg\mathsf{b}) = 0 \tag{25}$$

Being operations on determiners, outer negation, inner negation and dual can compose with one another. Based on the definitions given above, one can easily deduce the composition rules as summarized in Table 1.

Table 1: Composition of Outer Negation, Inner Negation and Dual

$[\![\neg(\neg D)]\!] = [\![D]\!]$	$[\![\neg(D^{\sim})]\!] = [\![D^d]\!]$	$[\![\neg(D^d)]\!] = [\![D^{\sim}]\!]$
$[\![(\neg D)^{\sim}]\!] = [\![D^d]\!]$	$[\![(D^{\sim})^{\sim}]\!] = [\![D]\!]$	$[\![(D^d)^{\sim}]\!] = [\![\neg D]\!]$
$[\![(\neg D)^d]\!] = [\![D^{\sim}]\!]$	$[\![(D^{\sim})^d]\!] = [\![\neg D]\!]$	$[\![(D^d)^d]\!] = [\![D]\!]$

Furthermore, we can also deduce a number of equivalences involving the negations and duals of determiners. For example, let D_1, D_2 be determiners, a, b be unary predicates, and r be a binary predicate. Then according to Keenan 2003, we have the following "Facing Negations" rule:

$$[\![D_1^{\sim}[a, \neg D_2(b)(r)]]\!] = 1 \Leftrightarrow [\![D_1[a, D_2(b)(r)]]\!] = 1 \tag{26}$$

We also have the following equivalence which involves the notions of inner negation, dual and predicate negation (in what follows, $\neg r$ represents the negation of the binary predicate r with the denotation $[\![\neg r]\!] = \mathtt{U} \times \mathtt{U} - [\![r]\!]$):

$$[\![D_1^{\sim}[a, D_2^d(b)(\neg r)]]\!] = 1 \Leftrightarrow [\![D_1[a, D_2(b)(r)]]\!] = 1 \qquad (27)$$

2.4. *Scope Dominance and Scopelessness*

Quantified statements containing two or more quantifiers are often ambiguous. For example, the following sentence has two readings: "direct scope" (DS) reading and "inverse scope" (IS) reading:

Example 2. Every boy is taller than a girl.

Under the DS reading, the above sentence means that "for every boy there is a (possibly different) girl such that he is taller than her". Under the IS reading, the above sentence means that "there is a particular girl such that every boy is taller than her". An interesting fact concerning the aforesaid ambiguity is that the IS reading entails the DS reading but not vice versa.

A number of scholars, including Westerståhl 1996, Ben-Avi and Winter 2004, Altman, Peterzil, and Winter 2005 and Altman and Winter 2005, have studied quantifier scope ambiguities and the entailment relation between different readings of an ambiguous quantified statement. Although their studies are concerned with quantifier scope ambiguities, their findings can in fact be reinterpreted as inference patterns between quantified statements with their subjects and objects interchanged and the binary predicate changed to its converse.

For example, if Example 2 is changed to the following sentence:

Example 3. A girl is shorter than every boy.

then the aforesaid entailment "IS reading ⇒ DS reading" can be reinterpreted as the following entailment: "Example 3 ⇒ Example 2"[5]. In the literature, this kind of entailment is an example of a phenomenon called "scope dominance". Let Q_1 and Q_2 be type $\langle 1 \rangle$ quantifiers. Then Q_1 is scopally dominant over Q_2 iff for any binary predicate r,

$$[\![Q_1[Q_2(r)]]\!] = 1 \Rightarrow [\![Q_2[Q_1(r^{-1})]]\!] = 1 \qquad (28)$$

5. As we have reinterpreted the entailment relation, we now forget about the IS reading of quantified statements, and treat all quantified statements as if they had only one reading, namely the DS reading.

Thus, the entailment "Example 3 ⇒ Example 2" is an example of the scope dominance of $I(a)$ over $A(b)$, where a and b are any unary predicates. This is the most important instance of scope dominance. It was widely known even before there was any systematic study on scope dominance. What the aforesaid scholars have achieved is that they have discovered other instances of scope dominance.

In case two type ⟨1⟩ quantifiers are scopally dominant over each other, we have the phenomenon of "scope independence". Thus, Q_1 and Q_2 are scopally independent iff for any binary predicate r,

$$[\![Q_1[Q_2(r)]]\!] = 1 \Leftrightarrow [\![Q_2[Q_1(r^{-1})]]\!] = 1 \tag{29}$$

Two important instances of scope independence are that between $A(a)$ and $A(b)$, and that between $I(a)$ and $I(b)$, where a and b are any unary predicates.

We next introduce a special scope behavior called "scopelessness" which is manifested in individual terms. In this paper, we will use letters at the end of the alphabet list, namely ..., x, y, z to represent individual terms (including constants and variables). Under GQT, individual terms can be lifted to type ⟨1⟩ quantifiers. To highlight this point, we will use capital letters ..., X, Y, Z to represent type-lifted individual terms. The denotation of type-lifted individual terms is given below: let x be an individual term, then

$$\text{For all } \mathtt{a} \subseteq \mathtt{U}, [\![X]\!](\mathtt{a}) = 1 \text{ iff } [\![x]\!] \in \mathtt{a} \tag{30}$$

As type ⟨1⟩ quantifiers, individual terms can appear in any position of a quantified statement where a general type ⟨1⟩ quantifier can appear. Hence, $X(r)$ is a unary predicate made up of a type ⟨1⟩ quantifier X plus a binary predicate r, while $X[I(a)(r)]$ is a quantified statement made up of a type ⟨1⟩ quantifier X plus a unary predicate $I(a)(r)$.

According to Zimmermann 1993, individual terms are scopeless. This means that we can interchange an individual term with another type ⟨1⟩ quantifier in a quantified statement with a binary predicate and get an equivalent statement, so long as the binary predicate is changed to its converse. For example, the statement $A[a, X(r^{-1})]$, which can also be written as $A(a)[X(r^{-1})]$ and means "Every a is taller than x", can be rewritten as $X[A(a)(r)]$, which means "x is shorter than every a" and is equivalent with the former statement.

Not only can an individual term interchange with another type ⟨1⟩ quantifier, but it can also interchange with the negation operator ¬ and yield an equivalent statement. For example, $\neg[X(a)]$, which means "It is not the case that x is an a", can be rewritten as $X(\neg a)$, which means "x is a non-a" and is equivalent with the former statement.

Note that an individual term cannot interchange with a determiner, although in some cases we may make use of the symmetry or contrapositivity of the determiner to rewrite the quantified statement and then interchange X with the type $\langle 1 \rangle$ quantifier thus resulted. For example, in $E[X(r^{-1}), b]$ and $A[X(r^{-1}), b]$, X cannot interchange with E or A. But by making use of the symmetry of E and the contrapositivity of A, we can rewrite these two statements as $E[b, X(r^{-1})]$ and $A[\neg b, X(\neg r^{-1})]$. We can then interchange X with $E(b)$ and $A(\neg b)$ to obtain $X[E(b)(r)]$ and $X[A(\neg b)(\neg r)]$, respectively.

2.5. *Syllogisms*

In classical logic, syllogisms refer to inferences consisting of two premises and one conclusion that must conform to a certain format characterized by the so-called "Figures" and "Moods". Moreover, there is also a special nomenclature for classical syllogisms, such as AAA-1, EIO-2, and so on. A description of the format and nomenclature of the classical syllogisms can be found in Pagnan 2012.

In this paper, we adopt a more liberal approach and consider syllogisms as inferences with at least two quantified statements as premises and one quantified statement as conclusion without imposing further restrictions on the format of the quantified statements. As pointed out in Pagnan 2012, in some cases, a syllogism may also have additional assumptions concerning the existence of some of the predicates, apart from the two premises. These existential assumptions can in fact be seen as additional premises.

We next introduce an operation on syllogisms, namely indirect reduction. Under this operation, the conclusion and one of the premises of a syllogism are negated and interchanged. Thus, if we represent a syllogism schematically as $p_1, p_2 \vdash p_3$, then it can be transformed under indirect reduction to either $p_1, \neg p_3 \vdash \neg p_2$ or $\neg p_3, p_2 \vdash \neg p_1$. This operation satisfies the following proposition.

PROPOSITION 5. Indirect reduction transforms valid syllogisms to valid ones, and invalid syllogisms to invalid ones.

Proof. Let $\alpha : p_1, p_2 \vdash p_3$ be a syllogism. By applying indirect reduction, we will obtain $\beta : p_1, \neg p_3 \vdash \neg p_2$ or $\gamma : \neg p_3, p_2 \vdash \neg p_1$.

(a) Suppose first that α is valid. This means that in any model W under which p_1 and p_2 are both true, p_3 must also be true. We now show that β is also valid. Consider any model W under which p_1 and $\neg p_3$ are both true. To show that $\neg p_2$ must also be true under W, we assume by way of contradiction that $\neg p_2$ is false, i.e. p_2 is true. Then, we would have p_1, p_2 and $\neg p_3$ all true. But by the preceding fact, since p_1 and p_2 are both true, then p_3 must also be true under W. Then we have p_3 and $\neg p_3$ both true under W, a contradiction. We have thus shown that β is valid. In a similar fashion, one can also show that γ is valid.

(b) Next suppose that α is invalid. Then there must exist a countermodel for α, that is a model W under which p_1 and p_2 are both true and p_3 is false, or in other words, p_1, p_2 and $\neg p_3$ are all true. But then W is also a countermodel for β and γ, and so β and γ are also invalid. □

3. The Derivation Method DMcr

3.1. *Description of DMcr*

In this section, we introduce a derivation method called DMcr for deriving valid relational syllogisms with comparative relations. We first provide a description of DMcr.

The general idea of DMcr is as follows: we first make substitution into a valid simple syllogism α so that it contains a binary predicate, and then simplify it and derive from it an immediate inference β. We then choose another valid simple syllogism γ such that one of its premises contains the same determiner as the one in the conclusion of β, and make substitution into γ so that one premise of γ is identical to the conclusion of β. The result is a relational syllogism that "combines" the two simple syllogisms α and γ.

DMcr consists of the following seven steps. Step (i): choose a valid simple syllogism with the following abstract form:

$$Q_1(b),\ Q_2(a)\ \vdash\ D_3(a,b) \tag{31}$$

where Q_1 and Q_2 are type $\langle 1\rangle$ quantifiers and D_3 is a conservative, right increasing and right positive determiner and substitute $a = X(r^{-1})$ and $b = D_4(d)(r)$, where X is an arbitrary individual variable and D_4 is a right increasing determiner, into the above to obtain the following:

$$Q_1[D_4(d)(r)],\ Q_2[X(r^{-1})]\ \vdash\ D_3[X(r^{-1}), D_4(d)(r)] \tag{32}$$

Step (ii): simplify the conclusion of the above to obtain the following:

$$Q_1[D_4(d)(r)],\ Q_2[X(r^{-1})]\ \vdash\ D_4[d, X(r^{-1})] \tag{33}$$

The above simplification is in fact based on the following immediate inference (by replacing the conclusion of (32) with the conclusion of the following inference):

$$D_3[X(r^{-1}), D_4(d)(r)] \vdash D_4[d, X(r^{-1})] \tag{34}$$

Step (iii): interchange X with the type $\langle 1 \rangle$ quantifiers in one of the premises and the conclusion of (33) to obtain the following:

$$Q_1[D_4(d)(r)],\ X[Q_2(r)] \vdash X[D_4(d)(r)] \tag{35}$$

Step (iv): from the above, derive the following immediate inference:

$$Q_1[D_4(d)(r)] \vdash A[Q_2(r), D_4(d)(r)] \tag{36}$$

Step (v): choose a valid simple syllogism where one premise is an A-statement with the form:

$$A(e, f),\ p_1 \vdash p_2 \tag{37}$$

where p_1 and p_2 are quantified statements, and substitute $e = Q_2(r)$ and $f = D_4(d)(r)$ into the above (this substitution is to make the first premise below identical to the conclusion of (36)) to obtain

$$A[Q_2(r), D_4(d)(r)],\ p'_1 \vdash p'_2 \tag{38}$$

where p'_1 and p'_2 are the effects of the above substitution on p_1 and p_2.

Step (vi): replace the first premise of (38) by the premise of (36) to obtain

$$Q_1[D_4(d)(r)],\ p'_1 \vdash p'_2 \tag{39}$$

Step (vii) (optional): do some or all of the following: (a) apply indirect reduction to the syllogism obtained thus far; (b) transform some or all of the quantified statements in the syllogism obtained thus far to equivalent ones, such as by making use of the equivalences (26), (27) or (29); (c) replace a premise p_2 of the syllogism obtained thus far with a quantified statement p_1 such that p_1 and p_2 are related by an immediate inference of the form $p_1 \vdash p_2$; (d) replace the conclusion p_1 of the syllogism obtained thus far with a quantified statement p_2 such that p_1 and p_2 are related by an immediate inference of the form $p_1 \vdash p_2$.

3.2. *Illustration of DMcr*

We now illustrate the use of DMcr by an example. In Step (i), we choose the valid classical syllogism AII-3 with the following form (note that in the following, the determiner in the conclusion, namely I, is a conservative, right increasing and right positive determiner and so satisfies the requirement of Step (i)):

$$A(c,b),\ I(c,a)\ \vdash\ I(a,b) \tag{40}$$

By instantiating D_4 as M (which is a right increasing determiner, thus satisfying the requirement of Step (i)) and substituting $a = X(r^{>-1})$ and $b = M(d)(r^{>})$ into the above, we obtain the following:

$$A[c, M(d)(r^{>})],\ I[c, X(r^{>-1})]\ \vdash\ I[X(r^{>-1}), M(d)(r^{>})] \tag{41}$$

In Step (ii), we simplify the conclusion of the above to obtain

$$A[c, M(d)(r^{>})],\ I[c, X(r^{>-1})]\ \vdash\ M[d, X(r^{>-1})] \tag{42}$$

As pointed out in Subsection 3.1, the above simplification is in fact based on the following immediate inference:

$$I[X(r^{>-1}), M(d)(r^{>})]\ \vdash\ M[d, X(r^{>-1})] \tag{43}$$

Note that the above immediate inference is intuitively valid and can be exemplified as follows (where d and $r^{>}$ are interpreted as "basketball players" and "be taller than" respectively):

Example 4. There is an individual who is taller than x and is shorter than most basketball players. Therefore, x is shorter than most basketball players.

In Step (iii), we interchange X with $I(c)$ in the second premise and with $M(d)$ in the conclusion of (42) to obtain

$$A[c, M(d)(r^{>})],\ X[I(c)(r^{>})]\ \vdash\ X[M(d)(r^{>})] \tag{44}$$

In Step (iv), we derive the following immediate inference from the above:

$$A[c, M(d)(r^{>})]\ \vdash\ A[I(c)(r^{>}), M(d)(r^{>})] \tag{45}$$

Again the above immediate inference is intuitively valid and can be exemplified as follows (where c, d and $r^{>}$ are interpreted as "runners", "basketball players" and "be taller than" respectively):

EXAMPLE 5. Every runner is shorter than most basketball players. Therefore, every individual that is shorter than some runner is shorter than most basketball players.

In Step (v), we choose the valid classical syllogism AAA-1 (note that the first premise of the following is an A-statement, satisfying the requirement of Step (v)):

$$A(e, f),\ A(g, e) \vdash A(g, f) \tag{46}$$

By substituting $e = I(c)(r^{>})$ and $f = M(d)(r^{>})$ into the above, we obtain the following:

$$A[I(c)(r^{>}), M(d)(r^{>})],\ A[g, I(c)(r^{>})] \vdash A[g, M(d)(r^{>})] \tag{47}$$

In Step (vi), we replace the first premise of the above by the premise of (45) to obtain

$$A[c, M(d)(r^{>})],\ A[g, I(c)(r^{>})] \vdash A[g, M(d)(r^{>})] \tag{48}$$

Without going through the optional Step (vii), we have already got a valid relational syllogism with comparative relation as given above. An exemplification of the above, with c, d, g and $r^{>}$ interpreted as "runners", "basketball players", "swimmers" and "be taller than" respectively, is Example 1 in Section 1.

Of course, we may as well go through Step (vii) to further derive other valid relational syllogisms. For example, we may do the following. First, by virtue of the immediate inference $A[g, M(d)(r^{>})] \vdash A[g, M(d)(r^{\geq})]$ (this is an application of Proposition 2), replace the conclusion of (48) with $A[g, M(d)(r^{\geq})]$ to obtain the following:

$$A[c, M(d)(r^{>})],\ A[g, I(c)(r^{>})] \vdash A[g, M(d)(r^{\geq})] \tag{49}$$

Second, apply indirect reduction by negating and interchanging the first premise and the conclusion of the above to obtain the following:

$$O[g, M(d)(r^{\geq})],\ A[g, I(c)(r^{>})] \vdash O[c, M(d)(r^{>})] \tag{50}$$

Third, by virtue of the immediate inference

$$I[c, A(g)(r^{>-1})] \vdash A[g, I(c)(r^{>})]$$

(this immediate inference is based on the scope dominance of $I(c)$ over $A(g)$), replace the second premise of the above with $I[c, A(g)(r^{>-1})]$ to obtain the following:

$$O[g, M(d)(r^{\geq})],\ I[c, A(g)(r^{>-1})] \vdash O[c, M(d)(r^{>})] \tag{51}$$

Finally, apply the equivalence (27) to the first premise of the above (by using $O^{\sim} = I$, $M^d = \geq_{\frac{1}{2}}$ and $\neg r^{\geq} = r^{>-1}$) to obtain the following:

$$I[g, \geq_{\frac{1}{2}}(d)(r^{>-1})],\ I[c, A(g)(r^{>-1})] \vdash O[c, M(d)(r^{>})] \qquad (52)$$

The above is also a valid relational syllogism with comparative relation and can be exemplified as follows (where c, d, g and $r^{>}$ are interpreted as above):

Example 6. A swimmer is taller than at least half of the basketball players. A runner is taller than every swimmer. Therefore, not every runner is shorter than most basketball players.

3.3. *Soundness of DMcr*

In this subsection, we will prove that DMcr is sound by showing that each step in DMcr is either validity-preserving or is based on a valid inference. Steps (i) and (v) of DMcr involve substitutions into valid simple syllogisms. These are obviously validity-preserving as substitution into a valid syllogism will always yield a valid syllogism.

Step (vii)(a) involves indirect reduction. This is validity-preserving because as pointed out in Proposition 5, indirect reduction will transform a valid syllogism to another valid syllogism.

Step (iii) involves interchange of X with a type $\langle 1 \rangle$ quantifier. As pointed out in Subsection 2.4, such an operation will give a statement that is equivalent with the original statement. Step (vii)(b) also involves transformation of quantified statements in a syllogism to equivalent ones. These two steps are both validity-preserving because equivalent statements have the same truth value. Replacing any statement in an inference with an equivalent one will not affect the validity of the inference.

Steps (vi) and (vii)(c) involve premise replacement which transforms the inference Σ, $p_2 \vdash p_3$ to the inference Σ, $p_1 \vdash p_3$, given that $p_1 \vdash p_2$, where Σ represents a (possibly empty) set of statements. To prove that this transformation is validity-preserving, suppose Σ, $p_2 \vdash p_3$ and $p_1 \vdash p_2$ are both valid inferences. This means that (a) if the members of Σ and p_2 are all true, then p_3 is true; and (b) if p_1 is true, then p_2 is true. Now assume that the members of Σ and p_1 are all true, then by (b), p_2 is true, and so by (a), p_3 is also true. We have thus shown that Σ, $p_1 \vdash p_3$ is a valid inference given the validity of Σ, $p_2 \vdash p_3$ and $p_1 \vdash p_2$, and so the transformation is validity-preserving.

Step (vii)(d) (as well as Step (ii)) involves conclusion replacement which transforms the inference $\Sigma \vdash p_1$ to the inference $\Sigma \vdash p_2$, given that $p_1 \vdash p_2$, where Σ represents a set of statements. This transformation is also validity-preserving and can be proved in a way analogous to the proof in the previous paragraph.

Before considering Steps (iv) and (ii), we have to introduce a lemma which relies on the Semantic Deduction Theorem in Propositional Logic, which states that if we have the entailment $(\llbracket p_1 \rrbracket = 1 \wedge \llbracket p_2 \rrbracket = 1) \Rightarrow \llbracket p_3 \rrbracket = 1$, then we have the entailment $\llbracket p_1 \rrbracket = 1 \Rightarrow \llbracket p_2 \rightarrow p_3 \rrbracket = 1$ (where p_1, p_2, p_3 are statements), as well as the close relation between implications and universal statements, which can be summarized as follows (where x is an arbitrary individual variable and X is the corresponding type-lifted quantifier):

$$\llbracket \forall x[X(a) \rightarrow X(b)] \rrbracket = 1 \text{ iff } \llbracket A(a,b) \rrbracket = 1 \tag{53}$$

The above is in fact the truth condition for the universal quantifier A. Intuitively, the above states that if the implication "If x is a, then x is b" is true for all x, then the universal statement "Every a is b" is also true, and vice versa.

Based on the above, we can formulate the following lemma.

LEMMA 1. Let a, b be unary predicates, x be an arbitrary individual variable (with X being the corresponding type-lifted quantifier) and p be a statement. If we have the entailment $(\llbracket p \rrbracket = 1 \wedge \llbracket X(a) \rrbracket = 1) \Rightarrow \llbracket X(b) \rrbracket = 1$, then we have the entailment $\llbracket p \rrbracket = 1 \Rightarrow \llbracket A(a,b) \rrbracket = 1$.

Proof. Suppose we have the entailment $(\llbracket p \rrbracket = 1 \wedge \llbracket X(a) \rrbracket = 1) \Rightarrow \llbracket X(b) \rrbracket = 1$. Then by the Semantic Deduction Theorem, we have the entailment $\llbracket p \rrbracket = 1 \Rightarrow \llbracket X(a) \rightarrow X(b) \rrbracket = 1$. Since x is an arbitrary variable, the implication $X(a) \rightarrow X(b)$ means that for any individual $\mathtt{x}$ in the domain, if $\mathtt{x}$ belongs to a, then $\mathtt{x}$ also belongs to b, which is what $\forall x[X(a) \rightarrow X(b)]$ means, which by (53) is equivalent to $A(a,b)$. We thus have the entailment $\llbracket p \rrbracket = 1 \Rightarrow \llbracket A(a,b) \rrbracket = 1$. □

We now consider Step (iv), which involves the derivation of the immediate inference (36) from (35). We show that if the inference in (35) is valid, then the immediate inference in (36) is also valid. So suppose (35) is valid, which means that we have the entailment $(\llbracket Q_1[D_4(d)(r)] \rrbracket = 1 \wedge \llbracket X[Q_2(r)] \rrbracket = 1) \Rightarrow \llbracket X[D_4(d)(r)] \rrbracket = 1$. But then the assumptions of Lemma 1 are satisfied (note that X is the type-lifted quantifier of an arbitrary variable x in (35)), and so we can invoke that lemma to conclude that we have the entailment $\llbracket Q_1[D_4(d)(r)] \rrbracket = 1 \Rightarrow \llbracket A[Q_2(r), D_4(d)(r)] \rrbracket = 1$. We have thus shown that (36) is valid.

We finally consider Step (ii), which involves the use of the immediate inference (34). We now prove the validity of this inference in the following proposition.

PROPOSITION 6. Let D_3 be a conservative, right increasing and right positive determiner, D_4 be a right increasing determiner, d be a unary predicate, r be a comparative relation and x be an individual term (with X being the corresponding type-lifted quantifier). Then $[\![D_3[X(r^{-1}), D_4(d)(r)]]\!] = 1 \Rightarrow [\![D_4[d, X(r^{-1})]]\!] = 1$.

Proof. Let $[\![D_3[X(r^{-1}), D_4(d)(r)]]\!] = 1$. Since D_3 is conservative and right increasing, by Proposition 3, there exists a witness set ω of $D_3[X(r^{-1})]$ such that $\omega \subseteq [\![D_4(d)(r)]\!]$. According to the definition of witness set (22), we also have $\omega \subseteq [\![X(r^{-1})]\!]$. Combining the above, we have

$$\omega \subseteq [\![D_4(d)(r)]\!] \cap [\![X(r^{-1})]\!] \tag{54}$$

Since D_3 is right positive, by Proposition 4, ω must be non-empty. Thus, by the above, there is an individual item u whose denotation is a member of ω such that $[\![u]\!] \in [\![D_4(d)(r)]\!]$ and $[\![u]\!] \in [\![X(r^{-1})]\!]$. By the former and (30) we have $[\![U[D_4(d)(r)]]\!] = 1$ which can be rewritten as

$$[\![D_4[d, U(r^{-1})]]\!] = 1 \tag{55}$$

By the latter, we have

$$[\![r(u, x)]\!] = 1 \tag{56}$$

Now by the transitivity of r, we have $([\![r(z, u)]\!] = 1 \wedge [\![r(u, x)]\!] = 1) \Rightarrow [\![r(z, x)]\!] = 1$, where z is an arbitrary individual variable. By (16) and (30), this entailment can be rewritten as $([\![Z[U(r^{-1})]]\!] = 1 \wedge [\![r(u, x)]\!] = 1) \Rightarrow [\![Z[X(r^{-1})]]\!] = 1$. But then the assumptions of Lemma 1 are satisfied (note that Z is the type-lifted quantifier of an arbitrary variable z), and so we can invoke that lemma to deduce the entailment $[\![r(u, x)]\!] = 1 \Rightarrow [\![A[U(r^{-1}), X(r^{-1})]]\!] = 1$. By (56), it follows that $[\![A[U(r^{-1}), X(r^{-1})]]\!] = 1$. By the denotation of the universal quantifier A, this is equivalent to $[\![U(r^{-1})]\!] \subseteq [\![X(r^{-1})]\!]$. Since D_4 is right increasing, we then have $[\![D_4[d, U(r^{-1})]]\!] = 1 \Rightarrow [\![D_4[d, X(r^{-1})]]\!] = 1$. By (55), it follows that $[\![D_4[d, X(r^{-1})]]\!] = 1$, which is the desired conclusion. □

Summarizing the above, we have proved that DMcr is sound.

4. Extensions of DMcr

In the previous section, we have introduced DMcr for deriving valid relational syllogisms with comparative relations. In this section, we discuss some possible ways to extend the power of DMcr by introducing some further tools which may be used in two of the seven steps of DMcr.

4.1. *Step (iv)*

Step (iv) involves derivation of an immediate inference (36) whose conclusion is an A-statement. This has restricted the subsequent choice of syllogisms in Step (v). However, we can relax this restriction by introducing existential assumptions.

Murphree discussed the dependence of validity of numerical syllogisms on existential assumptions in Murphree 1997. According to Murphree 1997, there are two types of existential assumptions: minimum assumptions and maximum assumptions. A minimum assumption asserts the minimum number of members of a unary predicate, like "There are at least n_1 a", which will be represented by $n_1 Iaa$. According to Murphree 1997, such an assumption allows us to make the following inference:

$$n_1 I(a,a),\ n_3 A(a,b)\ \vdash\ (n_1 - n_3) I(a,b) \tag{57}$$

In contrast, a maximum assumption asserts the maximum number of members of a unary predicate, like "There are at most n_2 a", which will be represented by $n_2 Eaa$. According to Murphree 1997, such an assumption allows us to make the following inference:

$$n_2 E(a,a),\ n_3 I(a,b) \vdash\ (n_2 - n_3) A(a,b) \tag{58}$$

What is interesting is that we can combine the two types of assumptions on different predicates. For example, suppose we have the minimum assumption $n_1 I(a,a)$ and the maximum assumption $n_2 E(b,b)$. Starting from $n_3 A(a,b)$, by (57), we may conclude $(n_1 - n_3) I(a,b)$. Then by the symmetry of I, we have $(n_1 - n_3) I(b,a)$. Then, by (58), we may further conclude $(n_2 - n_1 + n_3) A(b,a)$. The above inference can be summarized as follows:

$$n_1 I(a,a),\ n_2 E(b,b),\ n_3 A(a,b)\ \vdash\ (n_2 - n_1 + n_3) A(b,a) \tag{59}$$

Thus, by introducing a minimum assumption or minimum plus maximum assumptions, we can derive from the immediate inference (36) a new inference including a numerical determiner by using (57) or (59). This new inference will be our new output of Step (iv) (instead of (36)). The valid syllogism chosen for Step (v) can then be one including the numerical determiner in the new inference.

The new inference is valid because it is based on one of the inferences (57) and (59), which are valid according to Murphree 1997 (and whose validity can be proved by using the denotations of the numerical determiners given in (10) - (13)). Hence, DMcr remains sound if we extend Step (iv) as introduced above.

For illustration, let us reconsider the example discussed in Subsection 3.2 up to the point when we derived the immediate inference given in (45), whose conclusion is the A-statement $A[I(c)(r^>), M(d)(r^>)]$. Now by introducing a minimum assumption $3I[I(c)(r^>), I(c)(r^>)]$ and a maximum assumption $5E[M(d)(r^>), M(d)(r^>)]$ and by using (59), we obtain the following (remember that $A = 0A$):

$$\begin{aligned} & 3I[I(c)(r^>), I(c)(r^>)],\ 5E[M(d)(r^>), M(d)(r^>)],\ A[I(c)(r^>), M(d)(r^>)] \\ \vdash\ & 2A[M(d)(r^>), I(c)(r^>)] \end{aligned} \tag{60}$$

We can then incorporate the first two assumptions of the above as additional assumptions of (45), and replace the conclusion of (45) with the conclusion of the above to obtain the following:

$$\begin{aligned} & 3I[I(c)(r^>), I(c)(r^>)],\ 5E[M(d)(r^>), M(d)(r^>)],\ A[c, M(d)(r^>)] \\ \vdash\ & 2A[M(d)(r^>), I(c)(r^>)] \end{aligned} \tag{61}$$

The above is our new output of Step (iv) (instead of (45)). Then in Step (v), we can choose a valid syllogism such that one of its premises is a $2A$-statement. Suppose we now choose the following valid numerical syllogism studied by Murphree:

$$2A(e, f),\ 4I(g, e)\ \vdash\ 2I(g, f) \tag{62}$$

Then by following Steps (v) and (vi) of DMcr, we can derive the following valid relational syllogism:

$$\begin{aligned} & 3I[I(c)(r^>), I(c)(r^>)],\ 5E[M(d)(r^>), M(d)(r^>)],\ A[c, M(d)(r^>)], \\ & 4I[g, M(d)(r^>)]\ \vdash\ 2I[g, I(c)(r^>)] \end{aligned} \tag{63}$$

An exemplification of the above, with c, d, g and $r^>$ interpreted as “runners”, “basketball players”, “swimmers” and “be taller than” respectively, is given below:

EXAMPLE 7. There are at least 3 persons who are shorter than a runner. There are at most 5 persons who are shorter than most basketball players. Every runner is shorter than most basketball players. At least 4 swimmers are shorter than most basketball players. Therefore, at least 2 swimmers are shorter than a runner.

4.2. *Step (vii)*

Step (vii) involves replacement of one or more quantified statement in the syllogism obtained thus far with another statement by using equivalences or immediate inferences. One sort of immediate inferences that can be used in this Step is based on the concept of scope dominance introduced in Subsection 2.4. The most important instance of scope dominance is that of $I(a)$ over $A(b)$, which is valid for any domain. Thanks to the studies of Westerståhl 1996, Ben-Avi and Winter 2004, Altman, Peterzil, and Winter 2005 and Altman and Winter 2005, other instances of scope dominance have been identified. However, some of these are only applicable to finite domains, such as the scope dominance of $M(a)$ over $E(b)$. Thus, if we restrict the domain to a finite one, we will be able to make use of immediate inferences that are based on such instances of scope dominance in Step (vii).

In this subsection, we will introduce a special instance of scope dominance that is only valid when the domain is finite and the binary predicate is a comparative relation. This is the scope dominance of $A(b)$ over $I(a)$, where b is a non-empty predicate (referring to a predicate whose denotation is not an empty set). Here is an exemplification of this instance of scope dominance (Note that the following example contains an additional assumption which states the non-emptiness of b):

EXAMPLE 8. There is some boy in this class. Every boy in this class is shorter than a girl in this class. Therefore, a girl in this class is taller than every boy in this class.

Note that the finiteness of the domain, the use of a comparative relation and the non-emptiness of b are all crucial to the validity of the above inference. If we replace the finite domain consisting of "boys and girls in this class" by the infinite domain consisting of "odd and even numbers" (with "shorter / taller" being changed to "smaller / larger"), or replace the comparative relation "is shorter / taller than" by the general binary predicate "loves / is loved", then the above inference becomes invalid. Moreover, if there are no boys and girls in this class, then the second premise above is vacuously true but the conclusion is false.

The scope dominance of $A(b)$ over $I(a)$ over a finite domain in respect of a comparative relation subject to non-emptiness of b is in fact a special case of a more general result. But before presenting the result, we first have to introduce the following lemma.

LEMMA 2. Let $\mathtt{X}$ be a non-empty finite set and $r^{\geq}$ be a non-strict comparative relation. Then there exists $\mathtt{x} \in \mathtt{X}$ such that for all $\mathtt{z} \in \mathtt{X}$, $[\![r^{\geq}]\!](\mathtt{x}, \mathtt{z}) = 1$.

Proof. We prove by induction on $|\mathtt{X}|$. First let $|\mathtt{X}| = 1$ and $\mathtt{X} = \{\mathtt{x}\}$. Now for any $\mathtt{x}$, we must have $[\![r^{\geq}]\!](\mathtt{x}, \mathtt{x}) = 1$ because if $[\![r^{\geq}]\!](\mathtt{x}, \mathtt{x}) = 0$, then by Proposition 1(ii), we will have $[\![r^{\geq}]\!](\mathtt{x}, \mathtt{x}) = 1$, a contradiction. Since $\mathtt{x}$ is the only member of $\mathtt{X}$, we have thus proved the proposition for the case $|\mathtt{X}| = 1$.

Next assume that the proposition is true for the case $|\mathtt{X}| = n$, and consider the case $|\mathtt{X}| = n + 1$. Choose an arbitrary element $\mathtt{x}$ from $\mathtt{X}$ and define $\mathtt{Y} = \mathtt{X} - \{\mathtt{x}\}$. Then $|\mathtt{Y}| = n$. By the induction hypothesis, there exists $\mathtt{y} \in \mathtt{Y}$ such that for all $\mathtt{z} \in \mathtt{Y}$, $[\![r^{\geq}]\!](\mathtt{y}, \mathtt{z}) = 1$. Now consider $\mathtt{x}$ and $\mathtt{y}$. By Proposition 1(ii), we must have either (i) $[\![r^{\geq}]\!](\mathtt{x}, \mathtt{y}) = 1$ or (ii) $[\![r^{\geq}]\!](\mathtt{y}, \mathtt{x}) = 1$. In case (i), by transitivity of $r^{\geq}$, we have found an element $\mathtt{x} \in \mathtt{X}$ such that for all $\mathtt{z} \in \mathtt{X}$, $[\![r^{\geq}]\!](\mathtt{x}, \mathtt{z}) = 1$. In case (ii), we have also found an element $\mathtt{y} \in \mathtt{X}$ such that for all $\mathtt{z} \in \mathtt{X}$, $[\![r^{\geq}]\!](\mathtt{y}, \mathtt{z}) = 1$. Thus, the proposition is true for the case $|\mathtt{X}| = n + 1$. By the principle of mathematical induction, we have thus shown that the proposition is true for any finite set $\mathtt{X}$. □

We then state and prove the following proposition.

PROPOSITION 7. Over a finite domain, let D be a conservative, right increasing and right positive determiner, a be a unary predicate, b be a non-empty unary predicate and r be a comparative relation. Then we have

$$[\![D[b, I(a)(r)]]\!] = 1 \Rightarrow [\![I[a, D(b)(r^{-1})]]\!] = 1 \tag{64}$$

$$[\![A[b, D(a)(r)]]\!] = 1 \Rightarrow [\![D[a, A(b)(r^{-1})]]\!] = 1 \tag{65}$$

Proof. We adopt the following notation convention in the proof. If r is a strict comparative relation, then $r^{\geq}$ is the corresponding non-strict comparative relation. If r is already a non-strict comparative relation, then $r^{\geq} = r$.

(i) Let the assumptions of the proposition and the premise of (64), namely $[\![D[b, I(a)(r)]]\!] = 1$, be true. Since D is conservative and right increasing, by Proposition 3, there exists a witness set ω of $D(b)$ such that

$$\omega \subseteq [\![I(a)(r)]\!] \tag{66}$$

According to the definition of witness sets (22), we also have

$$[\![D]\!]([\![b]\!],\omega) = 1 \tag{67}$$

On the other hand, since D is right positive, by Proposition 4, ω is a non-empty set of the finite domain. Thus, by Lemma 2, there exists a $\mathtt{v} \in \omega$ such that for all $\mathtt{y} \in \omega$, $[\![r^{\geq}]\!](\mathtt{v},\mathtt{y}) = 1$. Since $\mathtt{v} \in \omega$, by (66), we have $\mathtt{v} \in [\![I(a)(r)]\!]$, which means that there exists a $\mathtt{u} \in [\![a]\!]$ such that $[\![r]\!](\mathtt{u},\mathtt{v}) = 1$. Combining the above, we have for all $\mathtt{y} \in \omega$, $[\![r]\!](\mathtt{u},\mathtt{y}) = 1$, whether r is strict or non-strict[6]. We have thus proved that $\omega \subseteq \{\mathtt{y} : [\![r]\!](\mathtt{u},\mathtt{y}) = 1\}$. Since D is right increasing, from this and (67), we have $[\![D]\!]([\![b]\!],\{\mathtt{y} : [\![r]\!](\mathtt{u},\mathtt{y}) = 1\}) = 1$. This means that $\mathtt{u} \in \{\mathtt{x} : [\![D]\!]([\![b]\!],\{\mathtt{y} : [\![r]\!](\mathtt{x},\mathtt{y}) = 1\}) = 1\}$, which can be rewritten as $\mathtt{u} \in [\![D(b)(r^{-1})]\!]$. But since $\mathtt{u} \in [\![a]\!]$, we have thus proved that $[\![I[a, D(b)(r^{-1})]]\!] = 1$, which is the desired conclusion of (64).

(ii) Let the assumptions of the proposition and the premise of (65), namely $[\![A[b, D(a)(r)]]\!] = 1$, be true. This implies $[\![b]\!] \subseteq [\![D(a)(r)]\!]$. Since $[\![b]\!] \neq \emptyset$, by Lemma 2, there exists a $\mathtt{v} \in [\![b]\!]$ such that for all $\mathtt{y} \in [\![b]\!]$, $[\![r^{\geq}]\!](\mathtt{v},\mathtt{y}) = 1$. Since $\mathtt{v} \in [\![b]\!]$, we have $\mathtt{v} \in [\![D(a)(r)]\!]$. This implies $[\![D]\!]([\![a]\!],\{\mathtt{x} : [\![r]\!](\mathtt{x},\mathtt{v}) = 1\}) = 1$. Since D is conservative and right increasing, by Proposition 3, there exists a witness set ω of $D(a)$ such that

$$\omega \subseteq \{\mathtt{x} : [\![r]\!](\mathtt{x},\mathtt{v}) = 1\} \tag{68}$$

According to the definition of witness sets (22), we also have

$$[\![D]\!]([\![a]\!],\omega) = 1 \tag{69}$$

Now (68) means that for all $\mathtt{w} \in \omega$, $[\![r]\!](\mathtt{w},\mathtt{v}) = 1$. Combining this with the fact concerning the elements of $[\![b]\!]$ concluded above, we can deduce that for all $\mathtt{w} \in \omega$ and for all $\mathtt{y} \in [\![b]\!]$, $[\![r]\!](\mathtt{w},\mathtt{y}) = 1$, whether r is strict or non-strict[7]. This fact can be written as $\omega \subseteq [\![A(b)(r^{-1})]\!]$. Since D is right increasing, combining this with (69), we have $[\![D[a, A(b)(r^{-1})]]\!] = 1$, which is the desired conclusion of (65). □

If we now instantiate D as I in (65), then we will have $[\![A[b, I(a)(r)]]\!] = 1 \Rightarrow [\![I[a, A(b)(r^{-1})]]\!] = 1$. This shows that the scope dominance of $A(b)$

6. In case $r = r^{>}$, then by Proposition 1(vi), we can deduce $[\![r^{>}]\!](\mathtt{u},\mathtt{y}) = 1$ from $[\![r^{>}]\!](\mathtt{u},\mathtt{v}) = 1$ and $[\![r^{\geq}]\!](\mathtt{v},\mathtt{y}) = 1$. In case $r = r^{\geq}$, then by Proposition 1(iii), we can deduce $[\![r^{\geq}]\!](\mathtt{u},\mathtt{y}) = 1$ from $[\![r^{\geq}]\!](\mathtt{u},\mathtt{v}) = 1$ and $[\![r^{\geq}]\!](\mathtt{v},\mathtt{y}) = 1$.

7. In case $r = r^{>}$, then by Proposition 1(vi), we can deduce $[\![r^{>}]\!](\mathtt{w},\mathtt{y}) = 1$ from $[\![r^{>}]\!](\mathtt{w},\mathtt{v}) = 1$ and $[\![r^{\geq}]\!](\mathtt{v},\mathtt{y}) = 1$. In case $r = r^{\geq}$, then by Proposition 1(iii), we can deduce $[\![r^{\geq}]\!](\mathtt{w},\mathtt{y}) = 1$ from $[\![r^{\geq}]\!](\mathtt{w},\mathtt{v}) = 1$ and $[\![r^{\geq}]\!](\mathtt{v},\mathtt{y}) = 1$.

over $I(a)$ over a finite domain in respect of a comparative relation subject to non-emptiness of b is a special case of Proposition 7.

The two immediate inferences in Proposition 7, which have been shown to be valid, enable us to transform a valid relational syllogism to another one which is also valid subject to certain additional assumptions (namely finite domain and non-emptiness of a predicate) in Step (vii). Hence, DMcr remains sound if we extend Step (vii) as introduced above.

For illustration, let us reconsider the relational syllogism discussed in Subsection 3.2 up to the point when Step (vi) is completed, but with D_4 instantiated as I (which is legitimate as I is right increasing, satisfying the requirement of Step (i) of DMcr) instead of M as in Subsection 3.2. Then we will obtain the following valid relational syllogism instead of (48) in Subsection 3.2:

$$A[c, I(d)(r^>)],\ A[g, I(c)(r^>)] \vdash A[g, I(d)(r^>)] \tag{70}$$

Then, in Step (vii), by virtue of the immediate inference $A[g, I(d)(r^>)] \vdash I[d, A(g)(r^{>-1})]$ subject to the additional assumption $I(g, g)$ (this is an application of (65)), we can replace the conclusion of the above with $I[d, A(g)(r^{>-1})$ and add the premise $I(g, g)$ to the above to obtain the following relational syllogism which is valid over a finite domain:

$$I(g, g),\ A[c, I(d)(r^>)],\ A[g, I(c)(r^>)] \vdash I[d, A(g)(r^{>-1})] \tag{71}$$

The above syllogism can be exemplified as follows (where c, d, g, $r^>$ are interpreted as "teachers of this class", "boys of this class", "girls of this class", "be taller than" respectively and the domain, namely "members of this class", is understood to be finite based on common world knowledge without being mentioned explicitly):

Example 9. There is some girl in this class. Every teacher of this class is shorter than a boy of this class. Every girl of this class is shorter than a teacher of this class. Therefore, a boy of this class is taller than every girl of this class.

If we remove the premise $I(g, g)$ from (71), then the syllogism is no longer valid. A countermodel is $[\![c]\!] = [\![d]\!] = [\![g]\!] = \emptyset$, because in this case we have $A[c, I(d)(r^>)]$ and $A[g, I(c)(r^>)]$ both true but $I[d, A(g)(r^{>-1})$ false.

Furthermore, if the domain is infinite, the syllogism (71) is also invalid. A countermodel is $[\![c]\!] = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$, $[\![d]\!] = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$, $[\![g]\!] = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$ and $r^>$ being the "bigger than" relation between integers, because in this case we have $I(g, g)$, $A[c, I(d)(r^>)]$ and $A[g, I(c)(r^>)]$ all true, but $I[d, A(g)(r^{>-1})$ false.

5. Conclusion

In this paper, we have presented a method called DMcr for deriving valid relational syllogisms with comparative relations and shown that it is sound. We have also discussed two possible ways to extend DMcr, including the use of existential assumptions, and restriction to finite domains, and shown that the extended method remains sound.

Unlike other logical systems which can only derive relational syllogisms with classical quantifiers, DMcr enables us to derive valid relational syllogisms with non-classical quantifiers by choosing valid simple syllogisms with non-classical quantifiers in Steps (i) and (v) and/or a suitable (namely right increasing) non-classical determiner as D_4 in Step (i). Thanks to the studies of a number of scholars, we now have an inventory of valid simple syllogisms with non-classical quantifiers, including "only" (studied by Dekker 2015), numerical quantifiers (studied by Murphree 1991, Murphree 1993, Murphree 1997, Murphree 1998), proportional quantifiers (studied by Johnson 1994), vague quantifiers (studied by Peterson 2000), to name just a few. Equipped with these simple syllogisms, DMcr is powerful in that it can derive relational syllogisms with a wide range of quantifiers (both classical and non-classical).

While we have proved the soundness of DMcr, we have not discussed its completeness. Given the great variety of relational syllogisms with comparative relations in terms of their syntactic structure as well as the types of quantifiers and predicates that they may contain, it is not a straightforward matter to determine, let alone prove, the completeness of DMcr. Therefore, the determination (and proof) of the completeness of DMcr will have to be left for future studies.

Before closing this paper, we would like to point to two possible directions for future studies on relational syllogisms. First, as pointed out in Section 1, by studying a special type of binary predicates, namely comparative relations, we can discover valid relational syllogisms that are not valid with general predicates. Apart from comparative relations, there are other types of special binary predicates. By studying these binary predicates, it is possible to discover still other valid relational syllogisms that are not valid with general predicates.

One special binary predicate is the equality relation "be identical to". According to the study of Pratt-Hartmann 2011, Hamiltonian syllogisms consisting of quantified statements of the form "D_1 a is D_2 b", where D_1, D_2 are classical quantifiers and a, b are unary predicates, can be seen as relational syllogisms with the equality relation, because it was asserted in Pratt-Hartmann 2011 that "D_1 a is D_2 b" should mean "D_1 a is identical to D_2 b".

Apart from the equality relation, in this paper we have also briefly discussed equative comparative relations, as exemplified by "be as tall as", without studying them in detail. Now equative comparative relations are related to but different from the equality relation as a person can be as tall as but not identical to another person. Thus, a possible theme for future study can be Hamiltonian syllogisms with non-classical quantifiers and relational syllogisms with equative comparative relations and the relationship between these two types of syllogisms.

Second, in Chow 2022 and this paper, we have only considered binary predicates. But in natural languages there are also ternary predicates as exemplified by the verb "give" and the preposition "between". In fact, some scholars, such as Murphree 1998 and Sommers and Englebretsen 2000, have mentioned some examples of relational syllogisms with ternary predicates. Like comparative relations, some ternary predicates such as "between" also satisfy a certain kind of "transitivity". For example, if we interpret $r(x, y, z)$ as "x is between y and z", then we have $(\llbracket r(x, y, z)\rrbracket = 1 \wedge \llbracket r(y, u, x)\rrbracket = 1) \Rightarrow \llbracket r(x, u, z)\rrbracket = 1$. Thus, a systematic study on relational syllogisms with ternary predicates, especially ternary predicates with special "transitivity" and other properties, will yield fruitful results.

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Ka-fat Chow
The Hong Kong Polytechnic University, HKSAR China
Email: kfzhouy@yahoo.com
ORCID iD: 0000-0002-5610-9342